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CERTAIN PERIODIC ORBITS OF k FINITE BODIES REVOLVING ABOUT A RELATIVELY LARGE CENTRAL MASS*

BY

FRANK LOXLEY GRIFFIN

§ 1. *The problem.*

Let a system of $k+1$ finite bodies be moving in a plane, subject to the newtonian law of attraction. Let the bodies be spheres, composed of homogeneous concentric layers; and let the mass of one of them, M, be large with respect to the masses of the others, $M_1, \cdots, M_k$. In other words, the distribution of masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. For convenience in what follows, the expression, planet and satellites, will be used, with the understanding that it covers also sun and planets.

Among all the possible variations of the configuration of the system, are there any which are periodic with an arbitrarily pre-assigned period, T; that is, do there exist initial conditions (probably dependent upon T and the masses) which result in periodic variations with the required period? It is proposed to discuss in this paper various cases in which periodic orbits of the k finite satellites exist, to show an easy method of constructing the solutions, and finally to make an application to the case of Jupiter's satellites I, II, and III. The fundamental method of analytical continuation of known orbits, due to POINCARÉ, is employed for the demonstrations of existence and convergence: for the construction, and elsewhere, certain variables and methods are used, which approximate to those employed by Professor MOULTON in his lectures on the lunar theory and in his recent contribution to the problem of three bodies.†

Let quantities $\mu, \beta_1, \cdots, \beta_k$, be defined thus:

$$M\beta_i\mu = M_i \qquad (i=1, \cdots, k), \tag{1}$$

where one of the β's is to be selected arbitrarily. Let a system of integers without common divisor, p_i, $(i=1, \cdots, k-1)$, and a number q_k be selected arbitrarily, save for one restriction mentioned below, and let the following defi-

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† *A class of periodic solutions of the problem of three bodies with an application to the lunar theory*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 537–577.

nitions be made:

$$\nu = \frac{2\pi}{T} = \frac{n_k}{q_k} = \frac{n_i - n_k}{p_i} \qquad (i=1, \cdots, k-1), \tag{2}$$

$$n_i^2 a_i^3 = \kappa^2 M \qquad (i=1, \cdots, k), \tag{3}$$

where κ^2 denotes the gravitational constant, and where, of the three values of a_i satisfying (3), that one is to be selected which is real. Also let the notation be so selected that $a_1, \cdots, a_k$ are in ascending order of magnitude, the p_i being so selected that no two of the a_i are equal.

If μ were zero — that is, if the satellites were "infinitesimal" — possible orbits would be conic sections about the planet with $a_1, \cdots, a_k$ as major semi-axes; from (3) it follows that the mean angular velocities would be $n_1, \cdots, n_k$. It is quite immaterial whether any of the n_i are negative: the results obtained hold, irrespective of retrograde motion of some of the bodies. In case all the eccentricities were zero, the configuration of the system would undergo periodic variations with period T; for it follows from (2) that

$$\frac{2\pi}{n_i - n_k} = \frac{T}{p_i},$$

or each synodic period is a sub-multiple of T. This condition being satisfied,* the motion of the system would be periodic with respect to a line through M, rotating with uniform angular velocity — that of M_k, or, indeed, that of any other body,† M_j — though whether or not the system ever returns to the same position in space depends upon whether q_k is rational or irrational.

[If, in some orbit, the eccentricity were not zero, then the necessary and sufficient conditions for periodicity are that each sidereal period be a sub-multiple of T, which requires that q_k be an integer, but permits common divisors for the p_i. This case is not treated in this paper.]

In describing the periodic orbits mentioned, the infinitesimal satellites would be subject to certain initial conditions; that is, at the instant $t = t_0$ the $2k$ coördinates and their derivatives with respect to the time would have certain

* POINCARÉ, in dealing with three satellites [*Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, pp. 154–6], with the apparent implication that his remarks apply equally to a larger number of bodies, formulates the condition thus: Integers, α, β, γ, mutually prime exist, such that

$$\alpha+\beta+\gamma=0, \qquad \alpha n_1+\beta n_2+\gamma n_3=0.$$

Evidently, in the case of three satellites, this condition is equivalent to (2), since $(n_1-n_3)/\beta=(n_2-n_3)/(-\alpha)$; but, for a greater number, it is not so. Thus, if $n_1=2\sqrt{2}$, $n_2=2$, $n_3=\sqrt{2}$, and $n_4=1$, the integers $2, -2, -1, 1$, satisfy a condition similar to POINCARÉ's; but periodicity is impossible. A re-formulation, such as (2), is necessary.

† The commensurability of n_i-n_k $(i=1, \cdots, k-1)$, evidently involves that of n_i-n_j $(i=1, \cdots, j-1, j+1, \cdots, k)$. For from $n_i-n_k=p_i\nu$ and $n_j-n_k=p_j\nu$ it follows that $n_i-n_j=(p_i-p_j)\nu$.

values, $c_{ij}(i=1,\cdots,k;\ j=1,\cdots,4)$. If the k finite satellites be subjected to the same initial conditions, the mutual disturbances, in general, destroy periodicity. But it is possible in various cases to vary the initial conditions (that is, let the initial values be $c_{ij}+\Delta c_{ij}$, where the Δc_{ij} are to be determined) so as to preserve the periodicity.

§ 2. *The differential equations.*

Let the common plane of motion of the $k+1$ bodies be selected as the ΞH-plane, the origin being at M, and $M\Xi$ and MH being rectangular axes which rotate in the plane with a uniform angular velocity, N. Let the coördinates of M_i referred to these axes be ξ_i and η_i; then the differential equations of motion are

$$(4)\quad (a)\ \frac{d^2\xi_i}{dt^2}-2N\frac{d\eta_i}{dt}-N^2\xi_i+\kappa^2(M+M_i)\frac{\xi_i}{r_i^3}+\sum_j{}'\kappa^2 M_j\left(\frac{\xi_i-\xi_j}{\rho_{ij}^3}+\frac{\xi_j}{r_j^3}\right)=0,$$

$$(b)\ \frac{d^2\eta_i}{dt^2}+2N\frac{d\xi_i}{dt}-N^2\eta_i+\kappa^2(M+M_i)\frac{\eta_i}{r_i^3}+\sum_j{}'\kappa^2 M_j\left(\frac{\eta_i-\eta_j}{\rho_{ij}^3}-\frac{\eta_j}{r_j^3}\right)=0,$$

where

$$r_i^2=\xi_i^2+\eta_i^2,\qquad \rho_{ij}^2=(\xi_j-\xi_i)^2+(\eta_j-\eta_i)^2,$$

and $\sum_j'$ means $\sum_{j=1}^{j=k}$, $(j\neq i)$.

Except in proving a certain symmetry theorem, these coördinates are less convenient than polar coördinates referred to rotating reference lines. Besides (1), (2) and (3), let the following definitions be made:

$$(5)\quad \begin{aligned} &\alpha_{ij}=a_i/a_j,\quad \nu q_i=n_i && (i=1,\cdots,k),\\ &\alpha_j\sigma_{ij}=\rho_{ij},\qquad \delta_{ij}=\beta_j q_1^2\alpha_{1i}\alpha_{1j} && (j\neq i),\\ &\nu t=\tau,\qquad \phi_{ji}=(p_j-p_i)\tau+(\lambda_j-\lambda_i), \end{aligned}$$

where the λ_i are arbitrary constants, later to be taken as the longitudes of the M_i at the origin of time, for $\mu=0$. Let polar coördinates be introduced by the equations

$$(6)\quad \begin{aligned} &\xi_i=r_i\cos u_i,\qquad \eta_i=r_i\sin u_i,\\ &r_i=a_i x_i,\qquad u_i=w_i+p_i\tau+\lambda_i, \end{aligned}$$

N being chosen equal to n_k, so that w_i is the longitude of M_i referred to a line rotating with uniform speed n_i. The differential equations become

$$(7)\quad (a)\ x_i w_i''+2x_i'(w_i'+q_i)+\mu\sum_j{}'\delta_{ij}x_j\sin(\phi_{ji}+w_j-w_i)\left(\frac{1}{x_j^3}-\frac{1}{\sigma_{ij}^3}\right)=0,$$

$$(7)\ (b)\quad x_i'' - x_i(w_i' + q_i)^2 + \frac{q_i^2(1+\beta_i\mu)}{x_i^2} + \mu \sum_j{}' \delta_{ij}\left[\frac{\alpha_{ij}x_i}{\sigma_{ij}^3} + x_{ij}\cos(\phi_{ji} + w_j - w_i)\left(\frac{1}{x_j^3} - \frac{1}{\sigma_{ij}^3}\right)\right] = 0,$$

where

$$a_j^2\sigma_{ij}^2 = a_i^2x_i^2 + a_j^2x_j^2 - 2a_ia_jx_ix_j\cos(\phi_{ji} + w_j - w_i),$$

and where the accents on the variables indicate derivatives with respect to τ.

§ 3. *Symmetry theorem.*

Definition. A *symmetrical conjunction* occurs when all k satellites are in a straight line through the planet, and are moving at right angles to the line.

THEOREM. *If a symmetrical conjunction occurs at an instant, $t = t_0$, then the orbit of each satellite before and after the conjunction is symmetrical, both with regard to geometric equality of figures and with regard to intervals of time.* The proof will be made only for the case $t_0 = 0$, which does not limit the generality since any other case is reduced to this one by the substitution $t = t_1 + t_0$. The differential equations (4) are invariant under the substitution

$$(8)\qquad \bar{\xi}_i = \xi_i, \qquad \bar{\eta}_i = -\eta_i, \qquad \bar{t} = -t.$$

Consequently every integral of (4) is transformed by (8) into some integral of (4). Moreover, the initial conditions

$$(9)\qquad \xi_i = a_i, \qquad \eta_i = 0, \qquad \frac{d\xi_i}{dt} = 0, \qquad \text{and} \qquad \frac{d\eta_i}{dt} = b_i,$$

at $t = 0$, are transformed into

$$(9')\qquad \bar{\xi}_i = a_i, \qquad \bar{\eta}_i = 0, \qquad \frac{d\bar{\xi}_i}{d\bar{t}} = 0, \qquad \text{and} \qquad \frac{d\bar{\eta}_i}{d\bar{t}} = b_i,$$

at $\bar{t} = 0$. Therefore, that solution of (4) which satisfies the initial conditions (9) is transformed by (8) into itself. Hence, if

$$(10)\qquad \xi_i = \Phi_i(t), \qquad \eta_i = \Psi_i(t), \qquad (i = 1, \cdots, k),$$

is that solution, then

$$\Phi_i(t) = \Phi_i(\bar{t}) = \Phi_i(-t), \qquad \Psi_i(t) = -\Psi_i(\bar{t}) = -\Psi_i(-t),$$

whence also

$$\xi_i(\eta_1, \cdots, \eta_k) = \xi_i(-\eta_1, \cdots, -\eta_k).$$

It will be noted that the proof holds whatever the value of N. It is geometrically evident that the symmetry, if present at all, is independent of the rate of rotation of the reference line.

§ 4. *Conditions for periodic solutions.*

Since the differential equations (7) are unchanged if τ be replaced by $\tau + 2n\pi$, or t by $t + nT$, (n being an integer), it follows that if

$$x_i = x_i(\tau), \qquad w_i = w_i(\tau) \qquad (i = 1, \cdots, k), \tag{11}$$

is a solution, then so is

$$x_i = x_i(\tau + 2n\pi), \qquad w_i = w_i(\tau + 2n\pi). \tag{12}$$

These two will be the same solution, if the coördinates and their derivatives have the same values at $\tau = \tau_0$; i. e., if

$$\begin{aligned} &x_i(\tau_0 + 2n\pi) = x_i(\tau_0), \quad w_i(\tau_0 + 2n\pi) = w_i(\tau_0), \\ &x_i'(\tau_0 + 2n\pi) = x_i'(\tau_0), \quad w_i'(\tau_0 + 2n\pi) = w_i'(\tau_0). \end{aligned} \tag{13}$$

If these conditions are satisfied, then, for all values of τ,

$$x_i(\tau + 2n\pi) = x_i(\tau), \qquad w_i(\tau + 2n\pi) = w_i(\tau);$$

that is, (13) are sufficient conditions for the periodicity of the solutions. That they are also necessary is obvious.

Special case. In the case of a symmetrical conjunction at $\tau = 0$, other sufficient conditions can be formulated. For, if $x_i'(0) = w_i(0) = 0$, ($i = 1, \cdots, k$), and if every λ_i is a multiple of π, it follows from the symmetry theorem that

$$\begin{aligned} &x_i(\pi) = x_i(-\pi), \qquad w_i(\pi) = -w_i(-\pi), \\ &x_i'(\pi) = -x_i'(-\pi), \quad w_i'(\pi) = w_i'(-\pi). \end{aligned} \tag{14}$$

But, by (13), if τ_0 be put equal to $-\pi$, the conditions for periodicity of $x_i(\tau)$ and $w_i(\tau)$ are

$$\begin{aligned} &(a) \quad x_i(\pi) = x_i(-\pi), \qquad (c) \quad w_i(\pi) = w_i(-\pi), \\ &(b) \quad x_i'(\pi) = x_i'(-\pi), \qquad (d) \quad w_i'(\pi) = w_i'(-\pi). \end{aligned} \tag{15}$$

Of these conditions, (a) and (d) are satisfied by virtue of (14); while (b) and (c) in connection with (14) require $x_i'(\pi) = w_i(\pi) = 0$. It may then be stated that sufficient conditions for the periodicity of x_i and w_i (with period in t equal to T), are

$$\begin{aligned} &(a) \quad x_i'(0) = 0, \qquad w_i(0) = 0, \qquad \lambda_i = 0, \pi, \\ &(b) \quad x_i'(\pi) = 0, \qquad w_i(\pi) = 0. \end{aligned} \tag{16}$$

For a symmetrical conjunction, when conditions (16a) are imposed, conditions (b) are necessary as well as sufficient.

§ 5. *Nature of the periodicity conditions.*

POINCARÉ has shown* that for a type of differential equations including (7) the solutions are developable as power series in μ and in the Δc_{ij} [see § 1], whose coefficients are functions of τ. And, for an arbitrarily assigned interval for τ, T_1 (herein taken greater than $2n\pi$), there exist values of μ and the Δc_{ij} sufficiently small (not zero), to make the series convergent for $0 \leqq \tau \leqq T_1$. Hence the conditions (13) and (16) are that certain power series in the $4k+1$ parameters, μ, Δc_{ij} $(i=1, \cdots, k;\ j=1, \cdots, 4)$ shall vanish.

Now, from the theory of implicit functions it is well known† that if (1) $f_i(y_1, \cdots, y_{4k};\ \mu)$ $(i=1, \cdots, 4k)$ are holomorphic in the vicinity of $\mu = y_j = 0$ $(j=1, \cdots, 4k)$, (2) $f_i(0, \cdots, 0;\ 0)=0$ $(i=1, \cdots, 4k)$, and if (3) the functional determinant (jacobian) of the f_i with respect to the y_j is distinct from zero for $\mu = y_j = 0$ $(j=1, \cdots, 4k)$, then there exist unique functions $y_j = y_j(\mu)$, which (a) are holomorphic in the vicinity of $\mu = 0$, (b) $f_i(y_1, \cdots, y_{4k};\ \mu)=0$, (c) vanish with μ. Even when (3) is not satisfied, there sometimes exist solutions having the properties (a), (b) and (c).

Let this theorem be applied to the conditions for periodicity, whose left-hand members, being convergent power series in μ and the Δc_{ij} and vanishing for $\mu = \Delta c_{ij} = 0$, evidently satisfy (1) and (2) above. The functional determinant, evaluated for $\mu = \Delta c_{ij} = 0$, is simply the determinant of the coefficients of the linear terms. In certain cases it vanishes and in others it does not. Under the latter circumstances, at least, periodic orbits exist, since the equations of condition define the Δc_{ij} as holomorphic functions of μ. For every value of μ sufficiently small there exist initial conditions [dependent upon T, μ, q_k, the β_i, and the p_i $(i=1, \cdots, k)$] such that the orbits described are periodic with the required period. It is evident that for smaller and smaller values of μ smaller and smaller deviations from the initial conditions of undisturbed motion are necessary in order to get periodic orbits. The periodic orbits for $\mu \neq 0$ may then be said to "grow out of" the undisturbed periodic orbits as μ grows from zero.

In this paper are discussed only those orbits which grow out of circles; and it is convenient to subdivide these into various types.

Type I. The system has a symmetrical conjunction.

Type II. The infinitesimal system has a symmetrical conjunction, but the finite system has not.

Type III. Neither system has a conjunction.

§ 6. *Integration of the differential equations as power series in parameters.*

It will be necessary to obtain the first few terms of the developments mentioned in the preceding section. Instead of increments Δc_{ij} to the initial

* Loc. cit., pp. 56–62.

† JORDAN, *Cours d'Analyse*, vol. 1, p. 82.

undisturbed values of the coördinates it will be more convenient, in finding the properties of the solutions, to employ parameters Δn_i, e_i, ω_i, τ_i, defined as follows. At $\tau = 0$ let

$$
\begin{aligned}
x_i = \quad & (1 + \Delta c_{i1}) && = (1 + \Delta n_i)^{-\frac{2}{3}}(1 - e_i \cos\theta_i), \\
\nu x_i' = \quad & \Delta c_{i2} && = n_i(1 + \Delta n_i)^{\frac{1}{3}} \frac{e_i \sin\theta_i}{1 - e_i \cos\theta_i}, \\
v_i - \lambda_i = \quad & w_i = \Delta c_{i3} && = \omega_i + \arccos \frac{\cos\theta_i - e_i}{1 - e_i \cos\theta_i} \qquad (i = 1, \cdots, k), \\
\nu w_i' = \quad & n_i \Delta c_{i4} && = \frac{n_i(1 + \Delta n_i)\sqrt{1 - e_i^2}}{(1 - e_i \cos\theta_i)^2} - n_i, \\
& - q_i(1 + \Delta n_i)\tau_i = \theta_i - e_i \sin\theta_i,
\end{aligned}
\tag{17}
$$

(the v_i, being the true longitudes from a fixed reference line, equal $u_i + q_k\tau$).

It is evident that the Δc_{ij} are holomorphic functions of the Δn_i, e_i, ω_i and τ_i for sufficiently small values of the latter quantities. Consequently, solutions of (7) exist also as power series in the new parameters. Further, since the real positive values of the radicals and the smallest values of the inverse cosines are to be taken in (17), the Δc_{ij} are given uniquely in terms of the Δn_i, e_i, ω_i and τ_i. From these two facts it follows that, if the latter quantities can be determined as unique power series in μ, satisfying the conditions for periodicity, then also there exist for the Δc_{ij} unique power series in μ, satisfying the conditions. Conversely, while the jacobian of the Δc_{ij} with respect to the new parameters is zero for $\Delta n_i = e_i = \omega_i = \tau_i = 0$, yet, in the only case where discussion will be necessary (viz. for $\Delta c_{i2} = \Delta c_{i3} = 0$, whence $\omega_i = \tau_i = 0$), the solution for the Δn_i and e_i in terms of the Δc_{i1} and Δc_{i4} is unique. For the jacobian of the Δc_{i1} and Δc_{i4} with respect to the Δn_i and e_i is distinct from zero for $\Delta n_i = e_i = 0$. Hence, in this case, if the Δc_{i1} and Δc_{i4} exist as unique series in μ, satisfying the periodicity conditions, so also must the Δn_i and e_i exist as such series.

In the developments of the coördinates as power series in μ and the new parameters all those terms independent of μ may be obtained, together with a knowledge of their properties, in the following simple manner. The terms in question are those remaining when μ is put equal to zero; and are, therefore, the solution of the problem of k infinitesimal satellites when the initial conditions are (17); in other words, the solutions of k two-body problems.

The dynamical meaning of the new parameters is then evident. The orbits of the infinitesimal system, subjected to initial conditions (17), are ellipses, in which the mean angular motions, major semi-axes, eccentricities, longitudes of pericenter, and times of pericenter passage are respectively

$$
n_i(1 + \Delta n_i), \qquad a_i(1 + \Delta n_i)^{-\frac{2}{3}}, \qquad e_i, \qquad \lambda_i + \omega_i, \qquad \text{and} \qquad \frac{\tau_i}{\nu}.
$$

If in the development of x_i the coefficient of $\Delta n_i^f e_i^g \omega_i^h \tau_i^k$ be denoted by $x_{i,fghk}$, then, by applying TAYLOR's theorem to the well known developments of the coördinates in elliptic motion as power series in the eccentricity,* it is found that the following coefficients of first and second degree terms do not vanish:

$$
\begin{aligned}
&x_{i,\,0000} = 1, && x_{i,\,1000} = -\tfrac{2}{3}, && x_{i,\,0100} = -\cos q_i\tau,\\
&x_{i,\,2000} = \tfrac{5}{9}, && x_{i,\,0200} = \sin^2 q_i\tau, && x_{i,\,0101} = -q_i \sin q_i\tau,\\
&x_{i,\,1100} = \tfrac{2}{3}\cos q_i\tau + q_i\tau \sin q_i\tau, && && w_{i,\,1000} = q_i\tau \qquad (18)\\
&w_{i,\,0100} = 2\sin q_i\tau, && w_{i,\,0010} = 1, && w_{i,\,0001} = q_i,\\
&w_{i,\,0200} = \tfrac{5}{4}\sin 2q_i\tau, && w_{i,\,0101} = -2q_i\cos q_i\tau, && w_{i,\,1100} = 2q_i\tau\cos q_i\tau.
\end{aligned}
$$

From simple dynamical considerations the following important properties may be established. Let $x_{i,fghk}$ be written $x_{i,fghk}^{(q\tau)}$ to indicate its dependence upon τ. Then

$$x_{i,\,0g00}^{\prime(m\pi)} = w_{i,\,0g00}^{(m\pi)} = 0 \qquad (i = 1, \cdots, k), \tag{19}$$

where m is an integer. For, the coefficients $x'_{i,\,0g00}$, etc., are those of the terms which do not involve Δn_i, ω_i and τ_i, these terms being obtained by putting $\Delta n_i = \omega_i = \tau_i = 0$, in the developments. But, for these parameters equal to zero, the initial positions are apses, and the periods (in t) are $2\pi/n_i$. Hence, at $\tau = m\pi/q_i$, M_i is at an apse; and $x'_i = w_i = 0$, whatever the value of e_i. Since this is true for a range of values of e_i, it follows that the coefficient of each power of e_i in x'_i and in w_i is zero.

It is evident that in the terms independent of μ appear only those parameters whose subscript is the same as that of the coördinate developed; the terms involving μ introduce, however, the other $4(k-1)$ parameters.

Terms involving μ.

The only terms involving μ whose coefficients are needed in the sequel are μ and $\mu e_j (j = 1, \cdots, k)$. Let the coefficient of μ in the development of x_i be $x_i(0\,;\,\tau)$, and that of μe_j be $x_i(j\,;\,\tau)$, let the coefficients of the same quantities in w_i be respectively $w_i(0\,;\,\tau)$ and $w_i(j\,;\,\tau)$, $(i = 1, \cdots, k\,;\, j = 1, \cdots, k)$. The process of finding these depends as follows upon two properties of the solutions.

(a) Since they must satisfy the differential equations identically in the parameters, the equating of coefficients of corresponding powers on both sides furnishes sets of differential equations for the successive coefficients in the solutions.

(b) The arbitrary constants which the successive coefficients carry are determined by the condition that the solutions shall reduce identically to (17) at $\tau = 0$.

* MOULTON, *Introduction to Celestial Mechanics*, pp. 154–5.

For each pair of coefficients, $x_i(f;\ \tau)$ and $w_i(f;\ \tau)$, $(f=0, \cdots, k)$, equations (7) give two simultaneous differential equations of the second order. The one from $(7a)$ can be integrated once immediately, and its integral combined with the equation from $(7b)$ renders the latter a well known type,

$$x_i''(f;\ \tau) + q_i^2 x_i(f;\ \tau) + \sum_{m=0}^{m=\infty} (\gamma_m \cos m\tau + \delta_m \sin m\tau) + \alpha\tau = 0. \tag{20}$$

Its solution, $x_i(f;\ \tau)$, when substituted into the first integral, permits the final integration for $w_i(f;\ \tau)$. The initial conditions are

$$x_i(f;\ 0) = x_i'(f;\ 0) = w_i(f;\ 0) = w_i'(f;\ 0) = 0 \qquad (i=1, \cdots, k\,;\ f=0, \cdots, k), \tag{21}$$

for the conditions (17) do not involve μ at all.

Now the form of the solution varies greatly according as a term $\cos q_i\tau$ or $\sin q_i\tau$ is or is not present in (20). In the former case the solutions contain a so-called Poisson term, $\tau \cos q_i\tau$, or $\tau \sin q_i\tau$, and in the latter case they do not. In all the $x_i(f;\tau)$ and $w_i(f;\tau)$, $(f=1, \cdots, k)$, a Poisson term is present; they are present in the $x_i(0;\tau)$, if, and only if, for some pair of the n_i, say n_f and n_g, there exists an integer, J, such that

$$J\cdot(n_f - n_g) = n_g. \tag{22}$$

The meaning and consequences of such a relation will be discussed in § 11.

In performing the integrations it is necessary to expand

$$(1 - 2\epsilon_{ij}\cos\phi_{ji} + \epsilon_{ij}^2)^{-s/2} \qquad (s=3, 5),$$

as a cosine series; where, for the sake of a uniform notation, the following definitions are made:

$$\begin{aligned} \epsilon_{ij} &= \alpha_{ji}, \quad \text{and} \quad \eta_{ij} = \alpha_{ji}, \quad \text{if } j<i, \\ \epsilon_{ij} &= \alpha_{ij}, \quad \text{and} \quad \eta_{ij} = 1, \quad \text{if } j>i. \end{aligned} \tag{23}$$

Also, where it is not essential to know the value of a Fourier sine or cosine series, it will be denoted simply by $F^s(\theta)$ or $F^c(\theta)$ respectively, where θ is the argument according to whose multiples the series proceeds. Now

$$\begin{aligned} (1 - 2\epsilon_{ij}\cos\phi_{ji} + \epsilon_{ij}^2)^{-\frac{3}{2}} &= \sum_{m=0}^{m=\infty} F_m(\epsilon_{ij})\cdot\cos m\phi_{ji}, \\ (1 - 2\epsilon_{ij}\cos\phi_{ji} + \epsilon_{ij}^2)^{-\frac{5}{2}} &= \sum_{m=0}^{m=\infty} G_m(\epsilon_{ij})\cdot\cos m\phi_{ji}, \end{aligned} \tag{24}$$

where the F_m and G_m are well known power series in ϵ_{ij}, beginning with ϵ_{ij}^m.

Finally, the desired coefficients are, for $i = 1, \cdots, k$,

$$
\begin{aligned}
x_i(0;\tau) &= A_{i0} + B_{i0}\cos q_i\tau + C_{i0}\sin q_i\tau + H_{i0}\tau\sin q_i\tau + \sum_j{}' F^e(\phi_{ji}),\\
w_i(0;\tau) &= D_{i0} - 2B_{i0}\sin q_i\tau + 2C_{i0}\cos q_i\tau\\
&\quad + 2H_{i0}\tau\cos q_i\tau + \sum_j{}' F^s(\phi_{ji}) - \frac{2H_{i0}}{q_i}\sin q_i\tau + E_{i0}\tau,
\end{aligned}
\tag{25}
$$

and for $f \neq i$,

$$
\begin{aligned}
x_i(f;\tau) &= A_{if} + B_{if}\cos q_i\tau + C_{if}\sin q_i\tau\\
&\quad + \delta_{if}L_{if}\tau\sin(q_i\tau + \lambda_i - \lambda_f) + \delta_{if}\{F^e(\phi_{fi}) + F^s(\phi_{fi})\},\\
w_i(f;\tau) &= D_{if} + E_{if}\tau - 2B_{if}\sin q_i\tau + 2C_{if}\cos q_i\tau\\
&\quad - \frac{2\delta_{if}L_{if}}{q_i}\sin(q_i\tau + \lambda_i - \lambda_f) + 2\delta_{if}L_{if}\tau\cos(q_i\tau + \lambda_i - \lambda_f)\\
&\quad + \delta_{if}\{F^s(\phi_{fi}) + F^e(\phi_{fi})\},
\end{aligned}
\tag{26}(A)
$$

while, for $f = i$,

$$
\begin{aligned}
x_i(i;\tau) &= A_{ii} + B_{ii}\cos q_i\tau + C_{ii}\sin q_i\tau + \sum_j{}'\delta_{ij}\{F^e(\phi_{ji}) + F^s(\phi_{ji})\}\\
&\quad + \tau\sin q_i\tau\cdot[H_{ii} + \sum_j{}'\delta_{ij}M_{ij}],\\
w_i(i;\tau) &= D_{ii} + E_{ii}\tau - 2B_{ii}\sin q_i\tau + 2C_{ii}\cos q_i\tau\\
&\quad + \sum_j{}'\delta_{ij}\{F^s(\phi_{ji}) + F^e(\phi_{ji})\} - \frac{2}{q_i}\sin q_i\tau\sum_j{}'\delta_{ij}M_{ij}\\
&\quad + 2\tau\cos q_i\tau\,[H_{ii} + \sum_j{}'\delta_{ij}M_{ij}].
\end{aligned}
\tag{26}(B)
$$

Of the coefficients and constants of integration, only the following are of importance:

$$
\begin{aligned}
L_{if} &= -\frac{\eta_{if}^3}{2q_i}\left[\tfrac{9}{2}F_0 + \tfrac{1}{4}F_2 - \tfrac{9}{2}\eta_{if}^2(1+\alpha_{if}^2)G_0 + \tfrac{11}{8}\alpha_{if}\eta_{if}G_1\right.\\
&\qquad \left. + \tfrac{3}{4}\eta_{if}^2(1+\alpha_{if}^2)G_2 - \tfrac{3}{8}\alpha_{if}\eta_{if}G_3\right],\\
M_{ij} &= \frac{\eta_{ij}^3}{2q_i}\left[\alpha_{ij}F_0 + 2F_1 - \alpha_{ij}\eta_{ij}^2(\tfrac{15}{2} + 3\alpha_{ij}^2)G_0 + 3\alpha_{ij}^2\eta_{ij}^2G_1 + \tfrac{9}{4}\alpha_{ij}\eta_{ij}^2G_2\right],
\end{aligned}
$$

(27) $H_{i0} \neq 0$, if, and only if, (22) holds,

$$
\begin{aligned}
C_{i0} &= \sum_j{}'\delta_{ij}\sum_{m=1}^{\infty}E_m(\epsilon_{ij})\sin m(\lambda_j - \lambda_i),\\
H_{ii} &= E_{i0} + \frac{1}{q_i}\sum_j{}'\delta_{ij}\eta_{ij}^3\left[\alpha_{ij}F_0(\epsilon_{ij}) - \tfrac{1}{2}F_1(\epsilon_{ij})\right],
\end{aligned}
$$

the $E_m(\epsilon_{ij})$ being certain linear functions of the $F_m(\epsilon_{ij})$. Thus,

$$
\begin{aligned}
q_i[m^2(q_j-q_i)^2-q_i^2]\,E_m(\epsilon_{ij}) &= m(q_j-q_i)\eta_{ij}^3(\alpha_{ij}F_m-\tfrac{1}{2}[F_{m+1}+F_{m-1}]) \\
&\quad -q_i\eta_{ij}^3(F_{m+1}-F_{m-1}), \quad \text{if } m\neq 1, \\
q_i[(q_j-q_i)^2-q_i^2]\,E_1(\epsilon_{ij}) &= (q_j-3q_i)+(q_j-q_i)\eta_{ij}^3(\alpha_{ij}F_1-\tfrac{1}{2}[F_2+2F_0]) \\
&\quad -q_i\eta_{ij}^3(F_2-2F_0).
\end{aligned} \tag{27'}
$$

§ 7. *Existence of periodic orbits. Type I.*

For all values of μ and the Δn_i, e_i, ω_i and τ_i, sufficiently small, the solutions of differential equations (7), with initial conditions (17), are given by series in those parameters, the first few coefficients of which have been tabulated in (18), (25) and (26).

For $\mu = 0$, a solution, with period 2π (or T in t), is

$$x_i = 1, \qquad w_i = 0; \tag{28}$$

whence

$$r_i = a_i, \qquad u_i = \lambda_i + p_i\tau,$$

where $\Delta n_i = e_i = \omega_i = \tau_i = 0$ $(i = 1, \cdots, k)$. For $\mu \neq 0$ these $4k$ parameters can be so determined that the solutions are still periodic with the required period.

Let $\omega_i = \tau_i = 0$, and $\lambda_i = 0, \pi$ $(i = 1, \cdots, k)$. The satellites have, then, at $\tau = 0$, a symmetrical conjunction; therefore, by (16), the necessary and sufficient conditions for periodicity are

$$
\begin{aligned}
&(a)\ \ 0 = \Delta n_i(q_i\pi) + e_i(2\sin q_i\pi) + \mu w_i(0\,;\,\pi) + \cdots, \\
&(b)\ \ 0 = \Delta n_i(0) + e_i(q_i\sin q_i\pi)\mu x_i'(0\,;\,\pi) + \Delta n_i e_i x'_{i,\,1100}(\pi) \\
&\qquad\qquad + \mu\sum_{f=1}^{f=k} e_f x_i'(f\,;\,\pi) + \cdots \quad (i=1,\cdots,k).
\end{aligned} \tag{29}
$$

Evidently these functions $w_i(\pi)$ and $x_i'(\pi)$ satisfy conditions (1) and (2) of § 5; it remains simply to examine condition (3). The jacobian of the $w_i(\pi)$ and $x_i'(\pi)$ with respect to the Δn_i and e_i, taken for $\Delta n_i = e_i = \mu = 0$, will be denoted by Δ_1, and has the value

$$\Delta_1 \equiv \pi^k \cdot \prod_{i=1}^{i=k} q_i^2 \sin(q_i\pi), \tag{30}$$

so that its vanishing depends upon the q_i. Since, from (2) and (5), $q_i = q_k + p_i$ $(i = 1, \cdots, k-1)$, it follows that Δ_1 vanishes if, and only if, the arbitrary q_k be selected as an integer.

Case I: *q_k is not an integer.* Since $\Delta_1 \neq 0$, condition (3) of § 5 is satisfied; hence, periodic orbits exist, the Δn_i and e_i (and, therefore, the Δc_{i1} and Δc_{i4})

existing as unique power series in μ. The presence of q_k and of the arbitrary integers p_i shows that for every value of μ (sufficiently small) and for every given set of mass-ratios, β_i/β_k there exist a k-fold infinitude of periodic orbits of this type.

Case II: q_k *is an integer.* Here $\Delta_1 = 0$. Nevertheless the jacobian of the $w_i(\pi)$ with respect to the Δn_i, taken at $\Delta n_i = e_i = \mu = 0$ is

$$\Delta_2 = \pi^k \prod_{i=1}^{i=k} q_i \neq 0. \tag{31}$$

[The only possibility for $q_i = 0$ is $p_i = -q_k$ which requires $n_i = 0$. The infinitesimal system would have one body, the major axis of whose orbit is infinite, or else the body moves in a straight line through the planet. Let such a selection for p_i be excluded.] Hence (29a) can be solved for the Δn_i as power series in the e_i and μ, converging for sufficiently small values of the latter quantities. Now, by (19), every term in (29a), (b) has either μ or some Δn_i as a factor; hence the solutions have the form

$$\Delta n_i = \mu \cdot P_i(e_j, \mu) \qquad (i=1, \cdots, k; j=1, \cdots, k). \tag{32}$$

If the power series (32) be substituted into (29b), the resulting series converge for sufficiently small values of μ and e_j, and contain μ as a factor. (This merely means that $\Delta n_i = \mu = 0$ satisfy the periodicity conditions whatever be the e_j, and is nothing new.) If μ is divided out,* relations are obtained among the e_j and μ, of the form

$$0 = x_i'(0;\pi) - e_i \frac{1}{q_i \pi} x'_{i,\,1100}(\pi) \cdot w_i(0;\pi) + \sum_{j=1}^{j=k} e_j x_i'(j;\pi) + \mu\{\ \}_i + \cdots. \tag{29b'}$$

At this point two questions of importance arise; viz. as to the vanishing of the $x_i'(0;\pi)$, and as to the vanishing of the determinant of the coefficients of the linear terms, e_j. By (25) and (27) every $x_i'(0;\pi)$ is zero unless the relation (22) holds for some pair of the n_i. When such a relation does hold, the equations (29b') are not satisfied by $e_j = \mu = 0$, so that solutions for the e_j in μ, vanishing with μ, do not exist. Hence, periodic orbits of Type I, "growing out of circular orbits" do not exist, if, for any n_f and n_g, $J(n_f - n_g) = n_g$, where J is an integer.

When no such relation exists,† the equations (29b') are satisfied by $e_j = \mu = 0$; it remains to examine the determinant, Δ_3, of the first degree terms in e_j. Δ_3 involves in each of its elements power series in the ϵ_{ij}, or a_i/a_j, a_j/a_i; and it is not known that there are no sets of values of the ϵ_{ij} for which $\Delta_3 = 0$. It can,

* It is precisely this step which makes the selection of the parameters Δn_i, e_i, especially advantageous.

† That q_k an integer does not involve the existence of such a relation is shown in § 11.

however, be shown that there is an infinite number (a range) of values for which the determinant is distinct from zero.

Let P_{if} be any element of Δ_3, the first subscript indicating the row and the second the column; then

$$
\begin{aligned}
&P_{if} = x_i'(f;\pi) = q_i(-1)^{q_i}\pi\delta_{if}L_{if}, \qquad f \neq i,\\
&P_{ii} = x_i'(i;\pi) - q_i(-1)^{q_i}w_i(0;\pi)\\
&\quad = q_i(-1)^{q_i}\pi\sum_j{}'\,\delta_{ij}[q_iM_{ij} + \eta_{ij}^3\{\alpha_{ij}F_0(\epsilon_{ij}) - \tfrac{1}{2}F_1(\epsilon_{ij})\}],
\end{aligned}
\tag{33}
$$

the simplification in P_{ii} resulting from the last equation of (27). The ϵ_{ij} depend upon ν, q_k, the β_i and the p_i, which were arbitrarily chosen. It will now be shown that for a fixed selection of the β_i, ν and q_k there is an infinite number of selections of the p_i (viz. all for which the ϵ_{ij} are "sufficiently small") for which $\Delta_3 \neq 0$. For convenience, let all the ϵ_{ij} be expressed in terms of a single parameter; thus,

$$
a_i = b_i^2\alpha^{2(k-i)}a_k \qquad (i=1,\cdots,k-1), \tag{34}
$$

where the b_i, ν and q_k (hence also a_k and n_k), are constants independent of α. Every element of Δ_3 is, then, a power series in α; for

$$
\epsilon_{ij} = \frac{b_j^2}{b_i^2}\alpha^{2(i-j)}, \quad \text{and} \quad \eta_{ij} = \frac{b_j^2}{b_i^2}\alpha^{2(i-j)}, \quad \text{if } j<i,
$$

$$
\epsilon_{ij} = \frac{b_i^2}{b_j^2}\alpha^{2(j-i)}, \quad \text{and} \quad \eta_{ij} = 1, \quad \text{if } j>i,
$$

$$
\begin{aligned}
&q_i = \frac{q_k}{b_i^3}\alpha^{3(i-k)}, \qquad P_{ii} = \alpha^{(6i+4-6k)}Q_{ii}(\alpha),\\
&P_{if} = \alpha^{(10i-4f-6k)}\cdot Q_{if}(\alpha), \qquad \text{if } f<i,\\
&P_{if} = \alpha^{(8f-2i)}\cdot Q_{if}(\alpha), \qquad \text{if } f>i,
\end{aligned}
\tag{35}
$$

where the $Q_{if}(\alpha)$ $(f=1,\cdots,k)$ are power series in α, *beginning with a constant term.* [In P_{ii} under the sign $\sum_j'$, the lowest power of α for $j<i$ is $(10i-4j-6k)$; for $j>i$, the lowest power is $6j-6k$. For all values of j taken in the summation, the lowest power is evidently $6i+4-6k$]. The lowest power of α in any of the P_{if} is then

$$
\begin{aligned}
&\text{for } f<i, \quad 6i+4-6k, \quad \text{viz. for } f=i-1,\\
&\text{for } f=i, \quad 6i+4-6k,\\
&\text{for } f>i, \quad 6i+8-6k, \quad \text{viz. for } f=i+1.
\end{aligned}
\tag{36}
$$

Hence, if the factor $\alpha^{6i+4-6k}$ be removed from the ith row $(i=1,\cdots,k)$, a

determinant Δ_4 is obtained such that $\Delta_4 \cdot \alpha^{-k(3k-7)} = \Delta_3$, and each of its elements is a power series in α. In the ith row of Δ_4 two elements, those in columns $i-1$ and i, have present a constant term in their series. The series of every other element begins with a power of α not lower than the fourth. Each term in the development of Δ_4 will be a power series in α, and the only term beginning with a constant is the main diagonal product. For every term must contain one, and only one, element from each row and column. To get a term containing a constant, one must select in the first row the element P_{11}; the only possible selection from the second row is then P_{22}, since P_{21} would be a second element from the first column. Continuing thus, the only selection possible in the ith row is P_{ii}. When expanded, Δ_4 can be re-arranged as a single power series in α, whose constant term (being merely the product of those in the main diagonal) is distinct from zero. Hence, for $\alpha = 0$, $\Delta_4 \neq 0$, and from this it follows that for all values of α, sufficiently small, $\Delta_4 \neq 0$. Also, Δ_4 being a power series, vanishes, if at all, at a finite number of points within the circle of convergence.

But Δ_3 can vanish only with Δ_4. Hence in general, even in the case where q_k is an integer, it is possible to solve (29b') for the e_i in terms of μ (vanishing with μ). The substitution of these solutions into (32) gives the Δn_i, as well as the e_i, as holomorphic functions of μ, for μ sufficiently small. Periodic orbits with the required period, therefore, exist.

In drawing this last conclusion, however, a point of some delicacy arises. The p_i are functions of α in the foregoing argument, and obviously the p_i are not integers (as the formulation of the problem requires that they be), for all values of α on any interval. The question arises as to whether there are, indeed, any values of α, "sufficiently small," for which the p_i are integers. From (35) and $p_i = q_i - q_k$, it follows that

$$p_i = q_k\left(\frac{1}{b_i^3 \alpha^{3(k-i)}} - 1\right) \qquad (i=1, \cdots, k-1). \tag{37}$$

The present discussion will be confined to exhibiting a selection of the b_i, such that there are an infinite number of values of α less than any assigned quantity, for each of which the p_i are integers without common divisor. Let the assigned value be α_0, and let the b_i be defined by

$$b_i = \frac{1}{\alpha_0^{k-i}} \sqrt[3]{\frac{q_k}{q_k + k - i}} \qquad (i=1, \cdots, k-1); \tag{38}$$

and consider (37) for $\alpha = \alpha_0 \sqrt[3]{1/n}$, where n is an integer. Evidently, since q_k is an integer, and since

$$p_i = (q_k + k - i) n^{k-i} - q_k \qquad (i=1, \cdots, k-1), \tag{39}$$

the p_i are integers. Consider the possibility of a common factor. If p_{k-1} and

p_{k-2} have a common factor, their difference has the same factor. Thus, if there is a factor common to $(q_k+1)n-q_k$ and $(q_k+2)n^2-q_k$, it is also a factor of $n^2+n(n-1)(q_k+1)$. Similarly, if p_{k-2} and p_{k-3} have a factor in common, it must be a factor of $n^3+n^2(n-1)(q_k+2)$. Hence, if p_{k-1}, p_{k-2}, and p_{k-3} have a common factor, it must divide $n^2[n+(n-1)(q_k+2)]$ and $n[n+(n-1)(q_k+1)]$; and if it be prime to n, it must divide $[n+(n-1)(q_k+2)]$ and $[n+(n-1)(q_k+1)]$. But if these two integers have a common factor, it must divide their difference, $n-1$. Now n and $n-1$ are mutually prime; consequently there is no factor of $(n-1)$ which divides $[n+(n-1)(q_k+1)]$, and hence no factor common to p_{k-1}, p_{k-2}, p_{k-3}, unless it divides n. Let n be chosen a prime number. There are an infinite number of primes; hence the p_i have the stated property.

The periodic solutions exist then, and might be obtained as series in μ alone (convergent for sufficiently small values), by substituting into the original series in μ and the Δn_i and e_i the values of the latter $2k$ parameters, as obtained in terms of μ from the periodicity conditions. A far more advantageous method is, however, available.

§ 8. *Method of construction. Type I.*

It has been shown that, for μ sufficiently small, there exist series

$$(40)\quad x_i(\tau)=1+\sum_{n=1}^{n=\infty}x_{i,n}(\tau)\cdot\mu^n,\quad w_i(\tau)=\sum_{n=1}^{n=\infty}w_{i,n}(\tau)\cdot\mu^n\qquad(i=1,\cdots,k),$$

which (a) converge for $0\leqq\tau\leqq 2\pi$, (b) satisfy differential equations (7), (c) satisfy $x_i'(0)=w_i(0)=0$ identically in μ, and (d) make $x_i(\tau+2\pi)-x_i(\tau)=w_i(\tau+2\pi)-w_i(\tau)=0$, identically in μ.

From (d) and (a) follows the permanent convergence of series (40). From (c) and (d) follow respectively

$$(41)\qquad x_{i,n}'(0)=w_{i,n}(0)=0,$$

$$(42)\quad x_{i,n}(\tau+2\pi)-x_{i,n}(\tau)=w_{i,n}(\tau+2\pi)-w_{i,n}(\tau)=0.$$

These equations (41) and (42) will determine the constants of integration arising at each step.

First order terms.

Since (40) must satisfy (7) identically in μ, $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$ must satisfy

$$(43)\quad \begin{aligned}&(a)\ w_{i,1}''+2q_ix_{i,1}'+\sum{}'\delta_{ij}\sin\phi_{ji}\left(1-\eta_{ij}^3\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right)=0,\\ &(b)\ x_{i,1}''-2q_iw_{i,1}'-3q_i^2x_{i,1}+q_i^2\beta_i\\ &\qquad+\sum{}'\delta_{ij}\left[\cos\phi_{ji}+\eta_{ij}^3(\alpha_{ij}-\cos\phi_{ji})\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right]=0.\end{aligned}$$

Since every $\phi_{j,i}$ is a multiple of τ, plus a multiple of π, equations (43) are of the type

$$(44)\quad (a)\qquad w''_{i,1} + 2q_i x'_{i,1} + \sum_{m=1}^{m=\infty} D^{(m)}_{i,1} \sin m\tau = 0,$$

$$(b)\quad x''_{i,1} - 2q_i w'_{i,1} - 3q_i^2 x_{i,1} + E^{(0)}_{i,1} + \sum_{m=1}^{m=\infty} E^{(m)}_{i,1} \cos m\tau = 0,$$

where the $D^{(m)}_{i,1}$ and $E^{(m)}_{i,1}$ are linearly related to the F_m, and can be expressed in terms of the latter as soon as the p_i are chosen. The solutions are

$$(45)\quad \begin{aligned} x_{i,1}(\tau) &= -\frac{1}{q_i^2}(2q_i c^{(1)}_{i,1} + E^{(0)}_{i,1}) + c^{(2)}_{i,1} \cos q_i\tau \\ &\qquad + c^{(3)}_{i,1} \sin q_i\tau + \sum_{m=1}^{\infty} A^{(m)}_{i,1} \cos m\tau, \\ w_{i,1}(\tau) &= \frac{1}{q_i}(3q_i c^{(1)}_{i,1} + 2E^{(0)}_{i,1})\tau + c^{(4)}_{i,1} - 2c^{(2)}_{i,1} \sin q_i\tau \\ &\qquad + 2c^{(3)}_{i,1} \cos q_i\tau + \sum_{m=1}^{\infty} B^{(m)}_{i,1} \sin m\tau, \end{aligned}$$

where the $c^{(j)}_{i,1}$ $(j = 1, \cdots, 4;\ i = 1, \cdots, k)$ are the constants of integration and

$$(46)\quad (m^2 - q_i^2) A^{(m)}_{i,1} = E^{(m)}_{i,1} - \frac{2q_i}{m} D^{(m)}_{i,1}, \qquad m^2 B^{(m)}_{i,1} = D^{(m)}_{i,1} - 2mq_i A^{(m)}_{i,1}.$$

Poisson terms do not appear in (45); for, since no relation (22) holds, no term in $\cos q_i\tau$ or $\sin q_i\tau$ is present in (44). Now by (41) and (42)

$$c^{(3)}_{i,1} = c^{(4)}_{i,1} = 0, \qquad c^{(1)}_{i,1} = -\frac{2}{3q_i} E^{(0)}_{i,1},$$

so that the k constants, $c^{(2)}_{i,1}$, alone remain to be determined. And here arise two cases, just as in the existence proof.

Case I. *q_k is not an integer.* Here, q_i not being an integer, $\cos q_i\tau$ does not have the period 2π; consequently by (42), $c^{(2)}_{i,1} = 0$ $(i = 1, \cdots, k)$.

Case II. *q_k is an integer.* Here $\cos q_i\tau$ has the period 2π, and (42) is satisfied for the arbitrary $c^{(2)}_{i,1} \neq 0$. These k constants remain undetermined until the second order terms are found, when the $c^{(2)}_{i,1}$ are uniquely determined in avoiding Poisson terms.

Terms of any order. Case I.

Assume that for $n = 1, \cdots, (h-1)$, $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have been found, the constants being determined, and have the form

$$(47)\quad x_{i,n}(\tau) = \sum_{m=0}^{m=\infty} A^{(m)}_{i,n} \cos m\tau, \qquad w_{i,n}(\tau) = \sum_{m=0}^{m=\infty} B^{(m)}_{i,n} \sin m\tau \quad (i = 1, \cdots, k).$$

An induction will show that $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ have the form (47), that, moreover, the differential equations for $x_{i,h}$ and $w_{i,h}$ are of the type (44), and that the constants of integration are determined just as in the case preceding, $n=1$. The differential equations are [see (7)]

$$(a)\quad \sum_{j+k=h} x_{i,j}w''_{i,k} + 2\sum_{j+k=h} x'_{i,j}w'_{i,k} + 2q_i x'_{i,h}$$

$$(48)\qquad + \sum_j{}' \delta_{ij} \sum_{k+l+m=h-1} x_{j,k}(\sin \overline{\phi_{ji} + w_j - w_i})_l [x_j^{-3} - \sigma_{ij}^{-3}]_m = 0,$$

$$(b)\quad x''_{i,h} - \sum_{j+k+l=h} x_{i,j}(w'_{i,k}w'_{i,l} + 2q_i w'_{i,k}) - q_i^2 x_{i,h} + q_i^2 (x_i^{-2})_h$$

$$+ q_i^2 \beta_i (x_i^{-2})_{h-1} + \sum_j{}' \delta_{ij}\{\alpha_{ij} \sum_{k+l=h-1} x_{i,k}(\sigma_{ij}^{-3})_l$$

$$+ \sum_{k+l+m=h-1} x_{j,k}(\cos \overline{\phi_{ji} + w_j - w_i})_l [x_j^{-3} - \sigma_{ij}^{-3}]_m \} = 0,$$

where an expression in a parenthesis, having a subscript, l, outside, denotes the sum of all those terms in the expression which involve μ^l. It is to be shown (α) that the variables in (48) whose second subscript is h enter in the same form as the $x_{i,1}$ and $w_{i,1}$ enter (44), and (β) that the remaining terms of (48) (a) and (b) reduce respectively to a sine series and a cosine series in multiples of τ.

Evidently in (48a) the only terms involving the $x_{i,h}$ and $w_{i,h}$ $(i=1,\cdots,k)$ are $w''_{i,h} + 2q_i x'_{i,h}$; and in (48$b$), aside from $(x_i^{-2})_h$, they are $x''_{i,h} - 2q_i w'_{i,h} - q_i^2 x_{i,h}$. Now it can be easily shown by induction that

$$\frac{d^h x_i^{-2}}{d\mu^h}\Big]_{\mu=0} = \sum_\nu N_\nu (-1)^{\nu_0} \prod_{f=0}^{f=h} x_i^{-(2+\nu_0)} \left(\frac{d^f x_i}{d\mu^f}\right)^{\nu_f}\Big]_{\mu=0},$$

where the N_ν are positive numbers and the ν_f are positive integers (or zero) satisfying the conditions

$$(49)\qquad \sum_{f=1}^{h} \nu_f = \nu_0, \qquad \sum_{f=1}^{h} f\nu_f = h.$$

Now $x_{i,h}$ enters only through $(d^h x_i/d\mu^h)^{\nu_h}$, and for this term $N_\nu = 2$, $\nu_0 = 1$, $\nu_h = 1$, and $\nu_f = 0$, $(f = 1, \cdots, h-1)$; hence, in $(x_i^{-2})_h$, $x_{i,h}$ appears with the coefficient -2. Therefore, in (48b), the terms involving $x_{i,h}$ and $w_{i,h}$ are $x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h}$, and this establishes statement (α) above.

Using again the notation $F^c(\tau)$ and $F^s(\tau)$ to designate respectively a cosine series and a sine series in multiples of τ, it is evident that

$$F^s_1 \cdot F^s_2 = F^c, \qquad F^c_1 \cdot F^c_2 = F^c, \qquad F^c \cdot F^s = F^s,$$

$$(F^c)^n = F^c, \qquad (F^s)^{2n} = F^c, \qquad (F^s)^{2n+1} = F^s.$$

Hence, as every $x_{i,n}$ and $w'_{i,n}$ $(n=1,\cdots,h-1)$ is a $F^c(\tau)$, so also are all sums of products of these quantities, and also all polynomials in the $x_{i,n}$, such

as $(x_i^{-\lambda})_l$ and $(\sigma_{ij}^{-3})_l$. Similarly the sums of all products $x_{i,j}w''_{i,k}$ and $x'_{i,j}w'_{i,j}$ are $F^s(\tau)$. There remain in (48) only the terms $(\sin m_{ij} + w_j - w_i)_l$ where $m_{ij} = \phi_{ji}$ in (48a), and $m_{ij} = \phi_{ji} + \pi/2$ in (48b). Let, for the moment, $z_{ij} = m_{ij} + w_j - w_i$. Then, since $[d^l z_{ij}/d\mu^l]_{\mu=0} = w_{j,l} - w_{i,l}$, it can be shown by a simple induction that

$$\left[\frac{d^l \sin z_{ij}}{d\mu^l}\right]_{\mu=0} = \sum_\nu N_\nu \prod_{f=0}^{l} \sin\left(m_{ij} + \nu_0 \frac{\pi}{2}\right)(w_{j,f} - w_{i,f})^{\nu_f},$$

where the N_ν are numbers and the ν_f satisfy (49) after h is replaced by l. Now since every $(w_{j,f} - w_{i,f})$, $(f = 1, \cdots, h-1)$ is a $F^s(\tau)$, the product $\Pi_{f=1}^{f=l}(w_{j,f} - w_{i,f})^{\nu_f}$ is a $F^s(\tau)$ or $F^c(\tau)$ according as $\sum_{f=1}^{f=l}\nu_f$ is odd or even. But when this sum is odd, ν_0 is odd; and when even, ν_0 is even. The entire product $\Pi_{f=0}^{f=l}$ is then always a $F^s(\tau)$ if $m_{ij} = \phi_{ji}$, or a $F^c(\tau)$ if $m_{ij} = \phi_{ji} + \pi/2$.

The differential equations (48) are, therefore, of the form

$$(50)\quad \begin{aligned} &(a)\quad w''_{i,h} + 2q_i x'_{i,h} + \sum_{m=1}^{m=\infty} D^{(m)}_{i,h} \sin m\tau = 0, \\ &(b)\quad x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h} + E^{(0)}_{i,h} + \sum_{m=1}^{m=\infty} E^{(m)}_{i,h} \cos m\tau = 0, \end{aligned}$$

where no term in $\cos q_i\tau$ or $\sin q_i\tau$ occurs under the summation sign, since q_i is not an integer. Obviously the integration of (50) is the same problem as that of (44), so that the solutions are

$$(51)\quad \begin{aligned} x_{i,h}(\tau) &= -\frac{1}{q_i^2}(2q_i c^{(1)}_{i,h} + E^{(0)}_{i,h}) + c^{(2)}_{i,h}\cos q_i\tau + c^{(3)}_{i,h}\sin q_i\tau + \sum_{m=1}^{m=\infty} A^{(m)}_{i,h}\cos m\tau, \\ w_{i,h}(\tau) &= \frac{1}{q_i}(3q_i c^{(1)}_{i,h} + 2E^{(0)}_{i,h})\tau + c^{(4)}_{i,h} - 2c^{(2)}_{i,h}\sin q_i\tau + 2c^{(3)}_{i,h}\cos q_i\tau \\ &\qquad + \sum_{m=1}^{m=\infty} B^{(m)}_{i,h}\sin m\tau \quad (i = 1, \cdots, k), \end{aligned}$$

where

$$(52)\quad (m^2 - q_i^2)A^{(m)}_{i,h} = E^{(m)}_{i,h} - \frac{2q_i}{m}D^{(m)}_{i,h}, \qquad m^2 B^{(m)}_{i,h} = D^{(m)}_{i,h} - 2mq_i A^{(m)}_{i,h}.$$

Also, by (41) and (42),

$$(53)\quad c^{(1)}_{i,h} = -\frac{2}{3q_i}E^{(0)}_{i,h}, \qquad c^{(2)}_{i,h} = c^{(3)}_{i,h} = c^{(4)}_{i,h} = 0 \qquad (i = 1, \cdots, k),$$

so that the induction is completely established. The coefficients of the successive $x_{i,n}$ and $w_{i,n}$ are obtained *without integration*, by simply applying (52) and the first equations of (53). It is merely necessary to compute at each step the $D^{(m)}_{i,h}$ and the $E^{(m)}_{i,h}$.

Terms of any order. Case II.

Assume, as in case I, that for $n = 1, \cdots, (h-1)$ the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have the form (47), all the constants of integration having been determined

with the exception of the $c^{(2)}_{i,h-1}$. The differential equations for the $x_{i,h}$ and $w_{i,h}$ are, then, (50a) and (b); where, however, since q_i is an integer, $\cos q_i\tau$ and $\sin q_i\tau$ may occur under the summation sign. These terms arise from two sources: from the terms $c^{(2)}_{i,h-1}\cos q_i\tau$ and $c^{(2)}_{i,h-1}\sin q_i\tau$ in the $x_{i,h-1}$ and $w_{i,h-1}$, and from similar terms in the earlier $x_{i,n}$ and $w_{i,n}$, as well as (usually) from combinations of the ϕ_{ji} in the coefficients.

When (50a) are integrated and combined with (50b), equations for the $x_{i,h}$ are obtained; to avoid Poisson terms in the $x_{i,h}$, the coefficients of the terms in $\cos q_i\tau$ in these last equations must be made to vanish. These coefficients, $E^{(q_i)}_{i,h}-D^{(q_i)}_{i,h}$, involve the $c^{(2)}_{i,h-1}$ and various known constants; e. g., the $c^{(2)}_{i,n}$ $(n=1,\cdots,h-2)$; and the vanishing of the $E^{(q_i)}_{i,h}-D^{(q_i)}_{i,h}$ will usually determine the $c^{(2)}_{i,h-1}$.

In the first place, the only terms of (48) in which the $x_{i,h-1}$ and $w_{i,h-1}$ appear are

$$
\begin{aligned}
(a)\quad & x_{i,h-1}w''_{i,1}+x_{i,1}w''_{i,h-1}+2x'_{i,h-1}w'_{i,1}+2x'_{i,1}w'_{i,h-1}\\
&\quad+\sum_j{}'\,\delta_{ij}\{[x_{j,h-1}\sin\phi_{ji}(1-\eta^3_{ij}\sigma^{-3}_{ij,0})]\\
&\quad+\sin\phi_{ji}[-3x_{j,h-1}-3\eta^5_{ij}\sigma^{-5}_{ij,0}(\alpha^2_{ij}x_{i,h-1}+x_{j,h-1}\\
&\quad-\alpha_{ij}\overline{x_{j,h-1}+x_{i,h-1}}\cos\phi_{ji}+\alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})]\\
&\quad+(1-\eta^3_{ij}\sigma^{-3}_{ij,0})\cos\phi_{ji}(w_{j,h-1}-w_{i,h-1})\},\\
(b)\quad & -2q_ix_{i,h-1}w'_{i,1}-2q_ix_{i,1}w'_{i,h-1}+6q_i^2x_{i,1}x_{i,h-1}\\
&\quad-2q_i^2\beta_ix_{i,1}x_{i,h-1}+\sum_j{}'\,\delta_{ij}\{[\alpha_{ij}x_{i,h-1}\eta^3_{ij}\sigma^{-3}_{ij,0}]\\
&\quad-3\eta^5_{ij}\sigma^{-5}_{ij,0}(\alpha_{ij}-\cos\phi_{ji})(\alpha^2_{ij}x_{i,h-1}+x_{j,h-1}\\
&\quad-\alpha_{ij}\overline{x_{j,h-1}+x_{i,h-1}}\cos\phi_{ji}+\alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\\
&\quad-\eta^3_{ij}\sigma^{-3}_{ij,0}[x_{j,h-1}\cos\phi_{ji}-\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji}]\\
&\quad-[2x_{j,h-1}\cos\phi_{ji}+\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji}]\},
\end{aligned}
\tag{54}
$$

where $\sigma_{ij,0}=(1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon^2_{ij})^{\frac{1}{2}}$. Hence it is evident that the $c^{(2)}_{f,h-1}$ enter the $E^{(q_i)}_{i,h}-D^{(q_i)}_{i,h}$ linearly; and, denoting their coefficients by R_{if} $(i=1,\cdots,k;\ f=1,\cdots,k)$, it is found that

$$
R_{if}=(-1)^{q_i}\frac{2}{\pi}P_{if}\qquad(i=1,\cdots,k;\ f=1,\cdots,k),
\tag{55}
$$

where the P_{if} are the elements of the determinant Δ_4, discussed in the existence proof. Hence, when $\Delta_4\neq 0$, so is the determinant of the coefficients of the $c^{(2)}_{f,h-1}$ in the equations $E^{(q_i)}_{i,h}-D^{(q_i)}_{i,h}=0$. The constants $c^{(2)}_{f,h-1}$ $(f=1,\cdots,k)$ are then uniquely determined.

The values of the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are given by (51) and (52); (53) are, however, replaced by

$$c_{i,h}^{(1)} = -\frac{2}{3q_i} E_{i,h}^{(0)}, \qquad c_{i,h}^{(3)} = c_{i,h}^{(4)} = 0 \qquad (i=1, \cdots, k), \tag{56}$$

the $c_{i,h}^{(2)}$ remaining undetermined at this step. The induction is thus established. In this case, too, the successive $x_{i,n}$ and $w_{i,n}$ are obtained *without integration* by merely computing the $D_{i,h}^{(m)}$ and $E_{i,h}^{(m)}$, applying (52) and the first equations of (56), and solving $E_{i,h+1}^{(q_i)} - D_{i,h+1}^{(q_i)} = 0$ for the $c_{i,h}^{(2)}$.

Remarks: 1. In constructing the solutions it has been tacitly assumed that the Fourier series representation of the $x_{i,n}$ and $w_{i,n}$, and the attendant manipulations of those series, are valid. This is justified by the consideration that the well-known functions encountered in the first step are representable, together with their derivatives, by uniformly convergent Fourier series. But the product of two such series is another of the same type, and also the integral of a uniformly convergent Fourier series is uniformly convergent. At every step the convergence remains uniform.

2. The question naturally arises as to whether, in any orbits of type I, the smallest period is a multiple of the period of the infinitesimal system, viz., mT. An examination of the method of constructing the solutions furnishes the answer. If mq_k is not an integer, the constants are determined precisely as in case I; if mq_k is an integer, precisely as in case II. In any event, for a given value of μ and a given set of p_i and q_k there is a *unique* solution satisfying (41) and (42) (with $2m\pi$ replacing 2π). The solutions found above, however, satisfy these conditions; hence there are no orbits whose smallest period is a multiple of T.

§ 9. *Orbits of type* II.

For type II, as for type I, all the λ_i are multiples of π; but it is proposed to ascertain whether the ω_i and τ_i exist as functions of μ (not identically zero, but vanishing with μ) so as to satisfy periodicity conditions (13). The question can be studied most easily by examining the method of construction.

Let the origin of time be selected as the instant when satellite k has an apsidal passage, and the origin of longitude at that satellite's apsidal position, so that $\omega_k = \tau_k = 0$; and let it be assumed for the moment that the ω_i and τ_i $(i = 1, \cdots, k-1)$, exist as functions of μ satisfying (13). Then there exist solutions of the type (40) satisfying (42), but not satisfying (41) for all values of n, except for $i = k$. Because of the absence of conditions (41) some of the constants in each step are left undetermined until terms of the next order are found.

For the first order terms the differential equations are again (44), and the solutions are (45), the coefficients being determined by (46). Evidently the $c_{i,1}^{(1)}$ are determined as in type I, but for the other $c_{i,1}^{(j)}$ two cases arise.

Case I. q_k *is not an integer.* Here, by (42), $c_{i,1}^{(2)} = c_{i,1}^{(3)} = 0$ $(i = 1, \cdots, k)$, but the $c_{i,1}^{(4)}$ are at present undetermined, except $c_{k,1}^{(4)} = 0$.

Case II. q_k *is an integer.* Here the terms $\cos q_i\tau$ and $\sin q_i\tau$ have the required period, 2π; and hence the $c_{i,1}^{(2)}$ and $c_{i,1}^{(3)}$, together with the $c_{i,1}^{(4)}$, remain undetermined, except $c_{k,1}^{(4)} = c_{k,1}^{(3)} = 0$.

Terms of any order. Case I.

Assume that the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ $(n = 1, \cdots, h-2)$ have the form (47), and that the $x_{i,h-1}(\tau)$ and $w_{i,h-1}(\tau) - c_{i,h-1}^{(4)}$ also have that form. Further, suppose all the constants, except the $c_{i,h-1}^{(4)}(i = 1, \cdots, k-1)$, to have been determined, the $c_{i,n}^{(4)}$ $(n = 1, \cdots, h-2)$ being zero.

The differential equations for the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are still (48); but, because of the $c_{i,h-1}^{(4)}$, these equations now reduce, not to (50), but to

$$(a) \qquad w_{i,h}'' + 2q_i x_{i,h}' + \sum_{m=0}^{m=\infty} (D_{i,h}^{(m)} \sin m\tau + H_{i,h}^{(m)} \cos m\tau) = 0,$$

(57)

$$(b) \quad x_{i,h}'' - 2q_i w_{i,h}' - 3q_i^2 x_{i,h} + \sum_{m=0}^{m=\infty} (E_{i,h}^{(m)} \cos m\tau + J_{i,h}^{(m)} \sin m\tau) = 0,$$

where the $H_{i,h}^{(m)}$ and $J_{i,h}^{(m)}$ vanish with the $c_{f,h-1}^{(4)}$. The first integration introduces non-periodic terms, $H_{i,h}^{(0)}\tau$. In order that the $x_{i,h}$ and $w_{i,h}$ may satisfy (42), these terms must disappear; their vanishing will determine the $c_{i,h-1}^{(4)}$.

In the ith equation of (57 a) let the coefficient of $c_{f,h-1}^{(4)}$ be K_{if}; then from (54 a) it is seen that the $c_{f,h-1}^{(4)}$ enter the $H_{i,h}^{(0)}$ linearly, and that

$$K_{if} = -\frac{\delta_{if}\eta_{if}^3}{2}\{F_1(\epsilon_{if}) + 3\alpha_{if}\eta_{if}^2[G_0(\epsilon_{if}) - \tfrac{1}{2}G_2(\epsilon_{if})]\} \qquad (f \neq i),$$

(58)

$$K_{ii} = -\sum_j{}' K_{ij} \qquad (i = 1, \cdots, k).$$

Since the $H_{i,h}^{(0)}$ vanish with the $c_{f,h-1}^{(4)}$, it follows that, if the $x_{i,h}$ and $w_{i,h}$ are to be periodic, the following linear equations must be satisfied:

$$\sum_{f=1}^{f=k-1} K_{if} c_{f,h-1}^{(4)} = 0 \qquad (i = 1, \cdots, k). \tag{59}$$

The necessary and sufficient condition that these k equations have a solution other than $c_{f,h-1}^{(4)} = 0$ $(f = 1, \cdots, k-1)$ is that every determinant of order $k-1$ formed from the matrix of k rows and $k-1$ columns should vanish. In this case, however, a more simple statement is possible. Since, if the summation in (59) extended from 1 to k, the determinant of all the coefficients would vanish identically (the sum of the elements of each row being zero), every solution of $k-1$ of the equations would satisfy the remaining one. Any solution of the

first $k-1$ equations of (59), taken with $c^{(4)}_{k,k-1}=0$, is, however, a solution of the first $k-1$ equations in which f runs from 1 to k; and, consequently, is a solution of the last equation of (59). Hence the necessary and sufficient condition that (59) have a solution, other than $c^{(4)}_{f,k-1}=0$, is that the determinant, Δ_5, of the coefficients in the first $k-1$ equations of (59) shall vanish.

It will here be shown merely that Δ_5 cannot vanish for any values of α [defined by (34)] on a certain interval, and that Δ_5 vanishes, if at all, only for a finite number of values of α. Since, by (58) and (35),

$$(60)\qquad \begin{aligned} K_{if} &= \alpha^{(10i-4f-6k)}\cdot R_{if}(\alpha), \qquad \text{if } f<i,\\ K_{if} &= \alpha^{(6f-6k)}\cdot R_{if}(\alpha), \qquad \text{if } f>i, \end{aligned}$$

where the R_{if} are power series in α beginning with a constant term, it follows that the lowest term in α in the ith row of Δ_5 ($i=1,\cdots,k-1$) is of degree $6i+4-6k$; and this power appears in the columns i and $i-1$. If, as in the case of Δ_3, the factor $\alpha^{(6i+4-6k)}$ be removed from the ith row ($i=1,\cdots,k-1$), another determinant, Δ_6, of order $(k-1)$ is obtained, such that $\Delta_6\cdot\alpha^{-(k-1)(3k-4)}=\Delta_5$, and each of its elements is a power series in α. The discussion of Δ_4 applies equally to Δ_6; and therefore Δ_5 vanishes, if at all, only for a finite number of values of α.

In general, then, there is only one solution of (59), viz., $c^{(4)}_{f,k-1}=0$ ($f=1,\cdots,k-1$); and since $c^{(4)}_{k,k-1}$ is also zero, the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ ($n=1,\cdots,k-1$) have the form (47). The differential equations for the $x_{i,k}(\tau)$ and $w_{i,k}(\tau)$ reduce, therefore, as in type I, to (50). The solutions are (51), where the constants are determined by (52) and (53), except the $c^{(4)}_{i,k}$ ($i=1,\cdots,k-1$) which are left undetermined. Hence, the $x_{i,k}(\tau)$ and $w_{i,k}(\tau)-c^{(4)}_{i,k}$ have the form (47), and the induction is established.

In general, then — if indeed not always — it is impossible in case I to determine the constants of integration otherwise than as in type I.

Terms of any order. Case II.

Assume that the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ ($n=1,\cdots,k-2$) have the form (47), and that the $x_{i,k-1}(\tau)-c^{(3)}_{i,k-1}\sin q_i\tau$ and the $w_{i,k-1}(\tau)-c^{(4)}_{i,k-1}-2c^{(3)}_{i,k-1}\cos q_i\tau$ also have that form. Suppose, further, that all the constants of integration have been determined, except $c^{(2)}_{k,k-1}$ and $c^{(j)}_{i,k-1}$ ($j=2,3,4$; $i=1,\cdots,k-1$), the $c^{(3)}_{i,n}$ and $c^{(4)}_{i,n}$ ($n=1,\cdots,k-2$) being zero.

The differential equations for the $x_{i,k}(\tau)$ and $w_{i,k}(\tau)$ are again (48), but because of the $c^{(3)}_{i,k-1}$ and $c^{(4)}_{i,k-1}$ these equations reduce to (57), where now the $H^{(m)}_{i,k}$ and $J^{(m)}_{i,k}$ vanish with the $c^{(3)}_{f,k-1}$ and $c^{(4)}_{f,k-1}$. In order that the $x_{i,k}(\tau)$ and $w_{i,k}(\tau)$ be periodic, it is necessary, just as in type I, that $E^{(q_i)}_{i,k}-D^{(q_i)}_{i,k}$, the coefficient of $\cos q_i\tau$, in the ith equation of the final set for the $x_{i,n}(\tau)$, should

vanish. It is equally necessary that $J_{i,k}^{(q_i)} - H_{i,k}^{(q_i)}$, the coefficients of $\sin q_i\tau$ in the same equation, should vanish. Finally, to avoid the "secular terms," $H_{i,k}^{(0)}\tau$, these coefficients must vanish. The vanishing of these three sets of coefficients will determine $c_{k,k-1}^{(2)}$ and the $c_{i,k-1}^{(j)}$ $(j=2,3,4;\ i=1,\cdots,k-1)$.

Now, from (54) and the fact that no relation (22) holds, it is evident that the $E_{i,k}^{(q_i)} - D_{i,k}^{(q_i)}$ do not involve the $c_{f,k-1}^{(3)}$ nor the $c_{f,k-1}^{(4)}$; also the $J_{i,k}^{(q_i)} - H_{i,k}^{(q_i)}$ involve neither the $c_{f,k-1}^{(2)}$ nor the $c_{f,k-1}^{(4)}$, and vanish with the $c_{f,k-1}^{(3)}$. The $H_{i,k}^{(0)}$ involve the $c_{f,k-1}^{(2)}$, the $c_{f,k-1}^{(3)}$, and the $c_{f,k-1}^{(4)}$. Further, since the $x_{i,k-1}(\tau)$ and $w_{i,k-1}(\tau)$ may be written

$$
\begin{aligned}
x_{i,k-1}(\tau) &= c_{i,k-1}^{(2)} \sin\left(q_i\tau + \frac{\pi}{2}\right) + c_{i,k-1}^{(3)} \sin q_i\tau + \sum_{m=0}^{m=\infty} A_{i,k-1}^{(m)} \cos m\tau, \\
w_{i,k-1}(\tau) &= 2c_{i,k-1}^{(2)} \cos\left(q_i\tau + \frac{\pi}{2}\right) + 2c_{i,k-1}^{(3)} \cos q_i\tau + \sum_{m=1}^{m=\infty} B_{i,k-1}^{(m)} \sin m\tau,
\end{aligned}
\tag{61}
$$

it is evident that the $c_{f,k-1}^{(3)}$ enter the $J_{i,k}^{(q_i)} - H_{i,k}^{(q_i)}$ (which are the coefficients of $\sin q_i\tau$), in precisely the same way as the $c_{f,k-1}^{(2)}$ enter the $E_{i,k}^{(q_i)} - D_{i,k}^{(q_i)}$, [which are the coefficients of $\sin(q_i\tau + \pi/2)$].

In the treatment of case II for type I it was shown that, when $\Delta_4 \neq 0$, the equations $E_{i,k}^{(q_i)} - D_{i,k}^{(q_i)} = 0$ admit a unique solution for the $c_{f,k-1}^{(2)}$ $(f=1,\cdots,k)$. The same conclusion is evidently valid here, and it follows at once that the equations

$$
J_{i,k}^{(q_i)} - H_{i,k}^{(q_i)} = 0 \qquad (i=1,\cdots,k), \tag{62}
$$

admit a unique solution for the $c_{f,k-1}^{(3)}$ $(f=1,\cdots,k)$ viz. $c_{f,k-1}^{(3)} = 0$. [In (62), $c_{k,k-1}^{(3)}$ does not appear, being zero; but this does not affect the conclusion, since there are obviously no more solutions for $k-1$ of the unknowns when the last is determined than there are for all k.]

From the manner in which the $c_{f,k-1}^{(2)}$ enter the $H_{i,k}^{(0)}$, it follows that, when the $c_{f,k-1}^{(3)}$ vanish, the $H_{i,k}^{(0)}$ reduce to the values they have in case I above. But, since equations (59) have in general no other solution $c_{f,k-1}^{(4)} = 0$ $(f=1,\cdots,k-1)$, there is in general only one way to determine the $c_{f,k-1}^{(j)}$ $(j=2,3,4;\ f=1,\cdots,k)$, viz. as in type I. The differential equations for the $x_{i,k}$ and $w_{i,k}$ reduce, therefore, to (50).

The solutions are given by (51) and (52), together with

$$
c_{k,k}^{(3)} = c_{k,k}^{(4)} = 0, \qquad c_{i,k}^{(1)} = -\frac{2}{3q_i} E_{i,k}^{(0)} \qquad (i=1,\cdots,k),
$$

the $c_{i,k}^{(j)}$ $(j=2,3,4;\ i=1,\cdots,k-1)$ and $c_{k,k}^{(2)}$ remaining undetermined. The $x_{i,k}(\tau) - c_{i,k}^{(3)} \sin q_i\tau$ and $w_{i,k}(\tau) - c_{i,k}^{(4)} - 2c_{i,k}^{(3)} \cos q_i\tau$ have the form (47); the induction is, then, established. Since at every step, as in case I, the

constants must be determined precisely as in type I, *there are in general no periodic solutions of type* II.

Remark. If either of the determinants Δ_3 and Δ_5 vanishes for some value of α, there is at least a single infinitude of solutions for the constants of integration, which would render the solutions periodic in form. Whether the series converge for this value of α is, however, unknown; so that the mere vanishing of Δ_3 or Δ_5 would not warrant the conclusion that orbits of type II exist.

§ 10. *Orbits of type* III.

It will next be inquired whether periodic orbits exist growing out of circular orbits of an infinitesimal system which has no "grand conjunction"; that is, whether there are periodic solutions of (7) when not all the λ_i are multiples of π. Let the initial conditions be (17); and let the origin of time be selected at an instant when M_k is at any apse, and the origin of longitude at the apsidal position of M_k, both for $\mu = 0$ and for $\mu \neq 0$. Then $\lambda_k = \omega_k = \tau_k = 0$.

Now the conditions for periodicity are (13). But equations (7) admit two integrals, an examination of which shows that two equations of (13), viz. $x_k(2\pi) = x_k(0)$, and $x'_k(2\pi) = x'_k(0)$, are a consequence of the other $4k-2$ equations. Hence equations (13) become

$$
\begin{aligned}
(a)\quad 0 &= e_i(1-\cos 2q_i\pi) + \mu\cdot X_{i,0} + e_i\tau_i(-\sin 2q_i\pi) \\
&\quad + \mu\sum_{f=1}^{f=k}\{\Delta n_f X^{(1)}_{i,f} + e_f X^{(2)}_{i,f} + \omega_f X^{(3)}_{i,f} + \tau_f X^{(4)}_{i,f}\} + \cdots, \\
(b)\quad 0 &= e_i(q_i\sin 2q_i\pi) + \mu X'_{i,0} + e_i\tau_i(q_i^2\overline{1-\cos 2q_i\pi}) \\
&\quad + \mu\sum_{f=1}^{f=k}\{\Delta n_f X'^{(1)}_{i,f} + e_f X'^{(2)}_{i,f} + \omega_f X'^{(3)}_{i,f} + \tau_f X'^{(4)}_{i,f}\} + \cdots \\
&\qquad\qquad (i=1,\cdots,k-1), \\
(63)\quad (c)\quad 0 &= \Delta n'_i(2q_i\pi) + e_i(2\sin 2q_i\pi) + \mu W_{i,0} + e_i\tau_i(2q_i\overline{1-\cos 2q_i\pi}) \\
&\quad + \mu\sum_{f=1}^{f=k}\{\Delta n_f W^{(1)}_{i,f} + e_f W^{(2)}_{i,f} + \omega_f W^{(3)}_{i,f} + \tau_f W^{(4)}_{i,f}\} + \cdots, \\
(d)\quad 0 &= e_i(-2q_i\overline{1-\cos 2q_i\pi}) + \mu W'_{i,0} + e_i\tau_i(q_i^2\sin 2q_i\pi) \\
&\quad + \mu\sum_{f=1}^{f=k}\{\Delta n_f W'^{(1)}_{i,f} + e_f W'^{(2)}_{i,f} + \omega_f W'^{(3)}_{i,f} + \tau_f W'^{(4)}_{i,f}\} + \cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
X_{i,0} &= x_i(0;2\pi) - x_i(0;0), \cdots, & W'_{i,0} &= w'_i(0;2\pi) - w'_i(0;0), \\
X^{(2)}_{i,f} &= x_i(f;2\pi) - x_i(f;0), \cdots, & W'^{(2)}_{i,f} &= w'_i(f;2\pi) - w'_i(f;0),
\end{aligned}
$$

and the $X^{(j)}_{i,f}$, $(j = 1, 3, 4)$, etc., are constants whose values will not be needed here.

While the determinant of the linear terms of the Δn_i, e_i, ω_i, and τ_i in (63) is zero; yet, when q_k is not an integer, solutions of (63) (c) and (d) exist for the Δn_i and e_i as power series in μ and the ω_i and τ_i. Properties of the series (63), similar to (19), are easily established, which show that the solutions for the Δn_i and e_i, and also the series obtained by substitution of these solutions into (63) (a) and (b), carry μ as a factor. After this substitution and a division by μ, (63) (a) and (b) become

$$(a)\quad 0 = X_{i,0} + \frac{1}{2q_i} W'_{i,0} + \sum_{f=1}^{f=k} \left\{ \omega_f \left[X^{(3)}_{i,f} + \frac{1}{2q_i} W'^{(3)}_{i,f} \right] + \tau_f \left[X^{(4)}_{i,f} + \frac{1}{2q_i} W'^{(4)}_{i,f} \right] \right\} + \cdots,$$

(64)

$$(b)\quad 0 = X'_{i,0} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'_{i,0} + \sum_{f=1}^{f=k} \left\{ \omega_f \left[X'^{(3)}_{i,f} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'^{(3)}_{i,f} \right] + \tau_f \left[X'^{(4)}_{i,f} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'^{(4)}_{i,f} \right] \right\} + \tau_i \frac{q_i W'_{i,0}}{1-\cos 2q_i\pi} + \cdots$$

The constant term in $(64a)$ vanishes, since, for q_k not an integer, no relation (22) holds; the constant term in $(64b)$ reduces to $-2q_i C_{i,0}$. If any of the $C_{i,0}$ are distinct from zero, (64) are not satisfied by $\omega_i = \tau_i = \mu = 0$; hence ω_i and τ_i do not exist as holomorphic functions of μ vanishing with μ, or periodic orbits of the type sought do not exist. The necessary condition for periodicity, viz., that the $C_{i,0}$ vanish, is, by (27),

$$\sum_j{}' \delta_{ij} \sum_{m=1}^{m=\infty} E_m(\epsilon_{ij}) \sin m(\lambda_j - \lambda_i) = 0 \qquad (i = 1, \cdots, k). \tag{65}$$

Of these k equations in the quantities λ_j $(j = 1, \cdots, k-1)$, one is evidently a consequence of the others; for, before λ_k is put equal to zero, the jacobian of the $C_{i,0}$ with respect to the λ_j $(i = 1, \cdots, k;\ j = 1, \cdots, k)$ is identically zero.

If particular values be assigned to all, save one, of the λ_j, the last λ can still be given an infinitude of values for which any one $C_{i,0}$ is distinct from zero; hence (65) impose very special conditions upon the λ_j. Whether there are any sets of λ's, other than multiples of π, which satisfy (65) is here left unsettled; it will, however, be shown that there are no others "in the vicinity of" multiples of π, provided α be sufficiently small.

If the $C_{i,0}$ are developed as power series in the $\lambda_f - J_f\pi$ $(J_f = 0, 1)$, the coefficients of the linear terms are simply $[\partial C_{i,0}/\partial\lambda_f]_{\lambda_i = J_i\pi}$. Evidently,

from (27),

$$(66)\qquad \left[\frac{\partial C_{i,0}}{\partial \lambda_f}\right]_{\lambda_i = J_i\pi} = \delta_{if}\sum_{m=1}^{\infty} mE_m(\epsilon_{ij})\cos m(J_f - J_i)\pi, \qquad f \neq i,$$

$$\left[\frac{\partial C_{i,0}}{\partial \lambda_i}\right]_{\lambda_i = J_i\pi} = -\sum_j{}' \delta_{ij}\sum_{m=1}^{\infty} mE_m(\epsilon_{ij})\cos m(J_j - J_i)\pi, \qquad f = i,$$

Denoting by S_{if} the coefficient of λ_f in the ith equation of (65), and introducing α by (34), the S_{if} are obtained as power series in α. From (27′) it is found that the lowest exponent of α present in S_{if} is $(f-i)$ if $f<i$, and $4(f-i)$ if $f>i$. Hence, of all the S_{if} $(f<i)$, S_{ii} carries the lowest power of α, viz., the $(1-i)$th; and of all the S_{if} $(f>i)$, $S_{i,i+1}$ carries the lowest, viz., the 4th. Consequently, if $\alpha^{(1-i)}$ be removed as a factor from all the S_{if} $(f=1,\cdots,k-1)$, a determinant Δ_7 is obtained which is equal to the determinant of the coefficients S_{if} multiplied by a power of α; and, in the ith row of Δ_7, all elements save those of the first and ith columns begin with a power of α. The discussion of Δ_4 applies also to Δ_7; showing that, for α sufficiently small, $\Delta_7 \neq 0$.

Since this determinant is distinct from zero, there exist quantities Λ_f $(f=1,\cdots,k-1)$ such that equations (65) are not satisfied for any values of the λ_f, for which $|\lambda_f - J_f\pi| < \Lambda_f$.

In considering the general question of the existence of any sets of the λ_f satisfying (65), if it be assumed for the moment that orbits of type III exist, and the method of construction be examined, equations related to (65) are encountered. Thus, in finding the second order terms, the $H^{(0)}_{i,2}$ must be made to vanish, and these involve the λ_f. In order that the k equations, $H^{(0)}_{i,2}=0$, be satisfied by the $k-1$ quantities $c^{(4)}_{i,1}$ $(i=1,\cdots,k-1)$, it is necessary that k determinants vanish, each of which involves Fourier series in the $\lambda_f - \lambda_i$.

When q_k is an integer, somewhat different difficulties arise in the existence proof; the same difficulty is, however, encountered in the construction. While a general negative conclusion is not yet warranted, it is evident that if any orbits of type III exist, they must satisfy very special conditions.

In every case, however, periodic orbits of the type sought do not exist if (22) holds.

§ 11. *Concerning lacunary spaces.*

The relation (22) may be expressed in the form $J(p_f - p_g) = p_g + q_k$; hence this relation can hold only if q_k is an integer. But the converse holds true only when $k=2$. For example, in the following selection q_k is an integer but no relation (22) holds:

$$q_k = 2, \qquad p_{k-1} = 3, \qquad p_{k-2} = 5, \cdots, \qquad p_1 = 2k-1.$$

When $k = 2$, (22) always holds* when q_k is an integer; for, since $q_2(p_1 - 0)/p_1 = 0 + q_2$, the relation holds if q_2/p_1 is an integer. If $p_1 \neq 1$, the period of the finite system is a multiple of that of the infinitesimal system; but no such orbits exist.

Since $n_k T = 2q_k\pi$, the case q_k an integer is the one where the consecutive conjunctions of the infinitesimal system occur at the same absolute longitude. And since, by taking T_{fg} to be the synodic period of the two infinitesimal satellites M_f and M_g,

$$n_g T_{fg} = n_g \frac{2\pi}{n_f - n_g},$$

it follows that, when (22) holds, the consecutive conjunctions of the infinitesimal pair M_f and M_g occur all in the same absolute longitude.

Moreover, since all the w_i vanish at the beginning and end of each period, all the "grand conjunctions" of the finite system occur at the same longitudes as those of the infinitesimal system, and intermediate conjunctions of any finite pair occur at very nearly the same longitudes as those of the corresponding infinitesimal pair.

These facts suggest a physical reason for the non-existence of periodic orbits under certain circumstances. The greater part of the mutual disturbances of two bodies occurs while they are near conjunction; and, if the consecutive grand conjunctions occur at exactly the same longitude, the perturbations of the elements would tend to be cumulative. Nevertheless, if there be more than two bodies, the mutual disturbances may so balance each other as to yield periodic orbits, especially if the bodies be far apart (i. e., α sufficiently small), unless (22) holds. But if two bodies have conjunctions between the grand conjunctions, all occurring very near the same longitude, the other bodies can not counterbalance the large perturbations of the two.

More exactly, there exists a range of values of the masses and the e_i, including zero, for which periodic orbits are impossible; so that, unless the orbits for $\mu = 0$ are eccentric rather than circular, there are for small values of μ no periodic plane orbits of k satellites when (22) holds.

So far as this result extends, it would indicate that no asteroids would be found, whose periods compared to Jupiter's are in the ratios $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, etc. Those whose periods are nearly in any such ratio should be found to be subject to very great perturbations.

It is well known that lacunary spaces of the sort just mentioned do occur among the asteroids. That there are such spaces also when the ratio of the periods is $\frac{3}{5}$, and other such values, is not surprising, as in such a case the

* In POINCARÉ's discussion of the problem of three bodies, therefore, the case where q_k is an integer and (22) does not hold can not arise.

slightest deviation from the correct initial values destroys periodicity, there being Poisson terms in the solutions.

§ 12. *Jupiter's satellites* I, II and III.

Of Jupiter's longer known satellites, the innermost three move almost exactly in a plane, apparently in periodic orbits having symmetrical conjunctions; and their masses with respect to that of the planet are very small. Since for orbits of type I the increase of the longitude of M_i during a period is independent of μ, being equal to $n_i T$, the average angular velocity of each satellite for a period may be taken as the corresponding n.

The unit of time being the sidereal day, and the unit of mass, the mass of Jupiter, the observational data are:*

$$\begin{aligned} &\lambda_1 = \pi, && \lambda_2 = 0, && \lambda_3 = 0, \\ &M_1 = 0.000017, && M_2 = 0.000023, && M_3 = 0.000088, \\ &n_1 = 203°.48895528, && n_2 = 101°.37472396, && n_3 = 50°.31760833. \end{aligned} \tag{67}$$

Since

$$\frac{n_1 - n_3}{3} = 51°.05711565, \qquad n_2 - n_3 = 51°.05711563,$$

it follows that the n_i of (67) satisfy, far beyond observational accuracy, the equations (2), where $\nu = 51°.05711564$. Taking arbitrarily $\mu = .0001$, one obtains

$$\begin{aligned} &\beta_1 = .17, && \beta_2 = .23, && \beta_3 = .88, \\ &p_1 = 3, && p_2 = 1, && q_3 = .985516077, \\ &\phi_{12} = \pi + 2\tau, && \phi_{13} = \pi + 3\tau, && \phi_{23} = \tau, \\ &\log \alpha_{12} = \bar{1}.79825923, && \log \alpha_{13} = \bar{1}.59545277, && \log \alpha_{23} = \bar{1}.79719354. \end{aligned} \tag{68}$$

It seems desirable, however, to retain the β's in the computations, inasmuch as a new determination of the masses may render it necessary to use other values than those given above. The δ_{ij} are then given by:

$$\begin{aligned} &\log \frac{\delta_{12}}{\beta_2} = 0.7974876, && \log \frac{\delta_{21}}{\beta_1} = 0.9992284, && \log \frac{\delta_{31}}{\beta_1} = 0.7964219, \\ &\log \frac{\delta_{13}}{\beta_3} = 0.3918747, && \log \frac{\delta_{23}}{\beta_3} = 0.1901339, && \log \frac{\delta_{32}}{\beta_2} = 0.3929403. \end{aligned} \tag{69}$$

Using the values of α_{ij} given in (68) the $F_m(\epsilon_{ij})$ and $G_m(\epsilon_{ij})$ are readily obtained from the tables of coefficients given by LEVERRIER.† To obtain the $D_{i,i}^{(m)}$ and

* TISSERAND, *Traité de Mécanique Céleste*, vol. 4, p. 2.

† Annales de l'Observatoire de Paris. (Mémoires) V. 2. Suppl.

$E^{(m)}_{i,1}$ of (44), equations (43) may be written

$$
\begin{aligned}
&w''_{i,1} + 2q_i x'_{i,1} + \sum_j{}' \delta_{ij} \sum_{m=1}^{\infty} U^{(m)}_{i,j} \sin m\phi_{ji} = 0, \\
&\qquad\qquad (i = 1, \cdots, k), \\
&x''_{i,1} - 2q_i w'_{i,1} - 3q_i^2 x_{i,1} + q_i^2 \beta_i + \sum_j{}' \delta_{ij} \sum_{m=0}^{\infty} V^{(m)}_{i,j} \cos m\phi_{ji} = 0,
\end{aligned} \tag{70}
$$

where

$$
U^{(1)}_{i,j} = 1 + \tfrac{1}{2}\eta^3_{i,j}(F_2 - 2F_0), \qquad U^{(m)}_{i,j} = \frac{\eta^3_{ij}}{2}(F_{m+1} - F_{m-1}) \quad (m \neq 1),
$$

$$
V^{(1)}_{i,j} = 1 + \tfrac{1}{2}\eta^3_{ij}[2\alpha_{ij}F_1 - (F_2 + 2F_0)], \qquad V^{(m)}_{i,j} = \frac{\eta^3_{ij}}{2}[2\alpha_{ij}F_m - (F_{m+1} + F_{m-1})] \quad (m \neq 1).
$$

Then: for $m =$ a prime, $D^{(m)}_{3,1} = U^{(m)}_{3,2}$ and $E^{(m)}_{3,1} = V^{(m)}_{3,2}$; $D^{(m)}_{2,1} = -U^{(m)}_{2,3}$, $E^{(m)}_{2,1} = V^{(m)}_{2,2}$; $D^{(m)}_{1,1} = E^{(m)}_{1,1} = 0$. Similarly, for $m = 2k$, where k is prime, $D^{(m)}_{1,1}$ and $E^{(m)}_{1,1}$ involve only $U^{(k)}_{1,2}$ and $V^{(k)}_{1,2}$; $D^{(m)}_{2,1}$ and $E^{(m)}_{2,1}$ involve both $U^{(k)}_{2,1}$ and $U^{(m)}_{2,3}$, $V^{(k)}_{2,1}$ and $V^{(m)}_{2,3}$; $D^{(m)}_{3,1}$ and $E^{(m)}_{3,1}$ involve only $U^{(m)}_{3,2}$ and $V^{(m)}_{3,2}$; etc.

In obtaining the successive $A^{(m)}_{i,n}$ and $B^{(m)}_{i,n}$ from (52), the smallest divisor introduced is $1 - q_3^2$, or 0.02875806. This divisor decreases materially the effectiveness of the small value of μ; nevertheless, the terms above those of the second order seem rather insignificant, and will not be computed here.

From the x_i given below the r_i are to be obtained by multiplying by the a_i, and from the w_i the v_i are obtained by adding the $n_i t$, whose values depend upon the values of the gravitational constant and of Jupiter's mass. The following values are found for the $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$:

$$
\begin{aligned}
x_{1,1}(\tau) = {} & (.33333\beta_1 - .07237\beta_2 - .01231\beta_3) - .44852\beta_2 \cos 2\tau - .08224\beta_3 \cos 3\tau \\
& - 94.43952\beta_2 \cos 4\tau + (.38582\beta_2 - .09643\beta_3) \cos 6\tau - .11181\beta_2 \cos 8\tau \\
& + .01380\beta_3 \cos 9\tau + .04361\beta_2 \cos 10\tau - (.01930\beta_2 + .00312\beta_3) \cos 12\tau \\
& + .00743\beta_2 \cos 14\tau + .00083\beta_3 \cos 15\tau - .00457\beta_2 \cos 16\tau \\
& + (.00235\beta_2 - .00024\beta_3) \cos 18\tau - .00124\beta_2 \cos 20\tau + .00007\beta_3 \cos 21\tau \\
& + .00067\beta_2 \cos 22\tau - (.00036\beta_2 + .00002\beta_3) \cos 24\tau \\
& + .00020\beta_2 \cos 26\tau + .00001\beta_3 \cos 27\tau - .00011\beta_2 \cos 28\tau \\
& + .00006\beta_2 \cos 30\tau - .00004\beta_2 \cos 32\tau + .00002\beta_2 \cos 34\tau \\
& - .00001\beta_2 \cos 36\tau + .00001\beta_2 \cos 38\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
w_{1,1}(\tau) = {} & - .73709\beta_2 \sin 2\tau - .10042\beta_3 \sin 3\tau - 94.32411\beta_2 \sin 4\tau \\
& + (.33625\beta_2 - .08575\beta_3) \sin 6\tau - .08890\beta_2 \sin 8\tau + .01088\beta_3 \sin 9\tau \\
& + .03248\beta_2 \sin 10\tau - (.01368\beta_2 + .00227\beta_3) \sin 12\tau + .00576\beta_2 \sin 14\tau \\
& + .00057\beta_3 \sin 15\tau - .00302\beta_2 \sin 16\tau + (.00152\beta_2 - .00016\beta_3) \sin 18\tau \\
& - .00078\beta_2 \sin 20\tau + .00005\beta_3 \sin 21\tau + .00041\beta_2 \sin 22\tau
\end{aligned}
$$

$$
\begin{aligned}
&- (.00022\beta_2 + .00001\beta_3) \sin 24\tau + .00012\beta_2 \sin 26\tau - .00007\beta_2 \sin 28\tau \\
&+ .00004\beta_2 \sin 30\tau - .00002\beta_2 \sin 32\tau + .00001\beta_2 \sin 34\tau \\
&- .00001\beta_2 \sin 36\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
x_{2,1}(\tau) = {} & (.49162\beta_1 + .33333\beta_2 - .07159\beta_3) - .49616\beta_3 \cos \tau \\
&- (65.88537\beta_1 + 54.22316\beta_3) \cos 2\tau - .42278\beta_3 \cos 3\tau \\
&+ (.47648\beta_1 - .11992\beta_3) \cos 4\tau - .04601\beta_3 \cos 5\tau \\
&- (.11925\beta_1 + .02011\beta_3) \cos 6\tau - .00946\beta_3 \cos 7\tau \\
&+ (.04236\beta_1 - .00467\beta_3) \cos 8\tau - .00238\beta_3 \cos 9\tau \\
&- (.01756\beta_1 + .00125\beta_3) \cos 10\tau - .00067\beta_3 \cos 11\tau \\
&+ (.00831\beta_1 - .00036\beta_3) \cos 12\tau - .00020\beta_3 \cos 13\tau \\
&- (.00382\beta_1 + .00011\beta_3) \cos 14\tau - .00006\beta_3 \cos 15\tau \\
&+ (.00191\beta_1 - .00004\beta_3) \cos 16\tau - .00002\beta_3 \cos 17\tau \\
&- (.00098\beta_1 + .00001\beta_3) \cos 18\tau - .00001\beta_3 \cos 19\tau \\
&+ .00052\beta_1 \cos 20\tau - .00028\beta_1 \cos 22\tau + .00015\beta_1 \cos 24\tau \\
&- .00008\beta_1 \cos 26\tau + .00005\beta_1 \cos 28\tau - .00003\beta_1 \cos 30\tau \\
&+ .00001\beta_1 \cos 32\tau - .00001\beta_1 \cos 34\tau + .00001\beta_1 \cos 36\tau \cdots.
\end{aligned}
$$

$$
\begin{aligned}
w_{2,1}(\tau) = {} & .83119\beta_3 \sin \tau + (64.53215\beta_1 - 54.05343\beta_3) \sin 2\tau - .35832\beta_3 \sin 3\tau \\
&- (.32599\beta_1 + .09204\beta_3) \sin 4\tau - .03302\beta_3 \sin 5\tau \\
&+ (.07104\beta_1 - .01374\beta_3) \sin 6\tau - .00623\beta_3 \sin 7\tau \\
&- (.02363\beta_1 + .00298\beta_3) \sin 8\tau - .00152\beta_3 \sin 9\tau \\
&+ (.00945\beta_1 - .00076\beta_3) \sin 10\tau - .00040\beta_3 \sin 11\tau \\
&- (.00425\beta_1 + .00021\beta_3) \sin 12\tau - .00012\beta_3 \sin 13\tau \\
&+ (.00173\beta_1 - .00006\beta_3) \sin 14\tau - .00004\beta_3 \sin 15\tau \\
&- (.00098\beta_1 + .00002\beta_3) \sin 16\tau - .00001\beta_3 \sin 17\tau \\
&+ (.00050\beta_1 - .00001\beta_3) \sin 18\tau - .00026\beta_1 \sin 20\tau \\
&+ .00014\beta_1 \sin 22\tau - .00008\beta_1 \sin 24\tau + .00004\beta_1 \sin 26\tau \\
&- .00002\beta_1 \sin 28\tau + .00001\beta_1 \sin 30\tau - .00001\beta_1 \sin 32\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
x_{3,1}(\tau) = {} & (.37879\beta_1 + .49038\beta_2 + .33333\beta_3) + 87.67296\beta_2 \cos \tau \\
&+ .52186\beta_2 \cos 2\tau - (.64483\beta_1 - .12770\beta_2) \cos 3\tau + .04461\beta_2 \cos 4\tau \\
&+ .01826\beta_2 \cos 5\tau + (.01205\beta_1 + .00819\beta_2) \cos 6\tau + .00389\beta_2 \cos 7\tau \\
&+ .00193\beta_2 \cos 8\tau - (.00224\beta_1 - .00099\beta_2) \cos 9\tau + .00052\beta_2 \cos 10\tau \\
&+ .00028\beta_2 \cos 11\tau + (.00053\beta_1 + .00015\beta_2) \cos 12\tau \\
&+ .00008\beta_2 \cos 13\tau + .00005\beta_2 \cos 14\tau - (.00014\beta_1 - .00003\beta_2) \cos 15\tau \\
&+ .00001\beta_2 \cos 16\tau + .00001\beta_2 \cos 17\tau + .00004\beta_1 \cos 18\tau \\
&- .00001\beta_1 \cos 21\tau + \cdots.
\end{aligned}
$$

$$w_{3,1}(\tau) = -85.53233\beta_2 \sin\tau - .34477\beta_2 \sin 2\tau - (.11320\beta_1 + .07379\beta_2)\sin 3\tau$$
$$- .02377\beta_2 \sin 4\tau - .00939\beta_2 \sin 5\tau - (.00535\beta_1 + .00413\beta_2)\sin 6\tau$$
$$- .00195\beta_2 \sin 7\tau - .00096\beta_2 \sin 8\tau + (.00099\beta_1 - .00049\beta_2)\sin 9\tau$$
$$- .00025\beta_2 \sin 10\tau - .00014\beta_2 \sin 11\tau - (.00023\beta_1 + .00007\beta_2)\sin 12\tau$$
$$- .00004\beta_2 \sin 13\tau - .00002\beta_2 \sin 14\tau + (.00006\beta_1 - .00001\beta_2)\sin 15\tau$$
$$- .00001\beta_2 \sin 16\tau - .00002\beta_1 \sin 18\tau + .00001\beta_1 \sin 21\tau + \cdots.$$

Since the second order terms carry the factor μ^2, they need not be obtained accurate to as many decimals; but almost as many multiples of τ are necessary.

The $x_{i,2}$ and $w_{i,2}$ [which are to be multiplied by .0001, if compared with the above terms] are:

$$x_{1,2}(\tau) = (-.1\beta_1^2 + 20.9\beta_1\beta_2 - .2\beta_1\beta_3 - 15.1\beta_2^2 + 12.2\beta_2\beta_3) - 60.4\beta_2\beta_3 \cos\tau$$
$$+ (687.2\beta_1\beta_2 - 85.3\beta_2^2 + 243.7\beta_2\beta_3)\cos 2\tau + (1.3\beta_1\beta_3 - .4\beta_2\beta_3$$
$$- 1.2\beta_3^2)\cos 3\tau + (49138.2\beta_1\beta_2 - 7844.0\beta_2^2 + 20059.1\beta_2\beta_3)\cos 4\tau$$
$$+ 10.3\beta_2\beta_3 \cos 5\tau - (228.2\beta_1\beta_2 - .6\beta_1\beta_3 + 22.5\beta_2^2 + 171.3\beta_2\beta_3)\cos 6\tau$$
$$+ 9.8\beta_2\beta_3 \cos 7\tau + (67.2\beta_1\beta_2 + 4475.5\beta_2^2 + 50.2\beta_2\beta_3)\cos 8\tau$$
$$- (.6\beta_1\beta_3 - 5.9\beta_2\beta_3 + .8\beta_3^2)\cos 9\tau - (27.9\beta_1\beta_2 + 4.0\beta_2^2 + 19.3\beta_2\beta_3)\cos 10\tau$$
$$+ 2.4\beta_2\beta_3 \cos 11\tau + (12.8\beta_1\beta_2 + 25.5\beta_2^2 + 14.2\beta_2\beta_3)\cos 12\tau$$
$$- 1.9\beta_2\beta_3 \cos 13\tau - (6.4\beta_1\beta_2 + 13.2\beta_2^2 + 8.3\beta_2\beta_3)\cos 14\tau$$
$$- .1\beta_2\beta_3 \cos 15\tau + (3.7\beta_1\beta_2 + 8.0\beta_2^2 + 4.9\beta_2\beta_3)\cos 16\tau + .2\beta_2\beta_3 \cos 17\tau$$
$$- (1.9\beta_1\beta_2 + 4.5\beta_2^2 + 2.4\beta_2\beta_3)\cos 18\tau - .3\beta_2\beta_3 \cos 19\tau + (1.0\beta_1\beta_2$$
$$+ 2.7\beta_2^2 + 1.2\beta_2\beta_3)\cos 20\tau - (.6\beta_1\beta_2 + 1.6\beta_2^2 + .6\beta_2\beta_3)\cos 22\tau$$
$$+ (.3\beta_1\beta_2 + 1.0\beta_2^2 + .4\beta_2\beta_3)\cos 24\tau - (.2\beta_1\beta_2 + .6\beta_2^2 + .3\beta_2\beta_3)\cos 26\tau$$
$$+ (.1\beta_1\beta_2 + .4\beta_2^2 + .2\beta_2\beta_3)\cos 28\tau - (.1\beta_1\beta_2 + .2\beta_2^2 + .1\beta_2\beta_3)\cos 30\tau$$
$$+ .2\beta_2^2 \cos 32\tau - .1\beta_2^2 \cos 34\tau + \cdots.$$

$$w_{1,2}(\tau) = 466.4\beta_2\beta_3 \sin\tau - (2409.0\beta_1\beta_2 - 271.2\beta_2^2 + 892.8\beta_2\beta_3)\sin 2\tau$$
$$- (3.0\beta_1\beta_2 - .9\beta_2\beta_3 - 3.3\beta_3^2)\sin 3\tau - (97989.6\beta_1\beta_2 - 15618.9\beta_2^2$$
$$+ 40019.6\beta_2\beta_3)\sin 4\tau - 7.0\beta_2\beta_3 \sin 5\tau + (351.8\beta_1\beta_2 - .9\beta_1\beta_3$$
$$+ 28.0\beta_2^2 + 262.6\beta_2\beta_3)\sin 6\tau + (.2\beta_2^2 - 18.6\beta_2\beta_3)\sin 7\tau - (89.6\beta_1\beta_2$$
$$+ 7818.4\beta_2^2 + 66.8\beta_2\beta_3)\sin 8\tau + (.6\beta_1\beta_3 - 5.4\beta_2\beta_3 + .7\beta_3^2)\sin 9\tau$$
$$+ (33.6\beta_1\beta_2 + 10.4\beta_2^2 + 19.2\beta_2\beta_3)\sin 10\tau - 2.8\beta_2\beta_3 \sin 11\tau$$
$$- (14.4\beta_1\beta_2 + 34.2\beta_2^2 + 16.3\beta_2\beta_3)\sin 12\tau + 2.8\beta_2\beta_3 \sin 3\tau$$
$$+ (7.1\beta_1\beta_2 + 17.2\beta_2^2 + 9.1\beta_2\beta_3)\sin 14\tau + .1\beta_2\beta_3 \sin 15\tau - (3.9\beta_1\beta_2$$
$$+ 9.5\beta_2^2 + 5.5\beta_2\beta_3)\sin 16\tau - .3\beta_2\beta_3 \sin 17\tau + (2.0\beta_1\beta_2 + 5.1\beta_2^2$$
$$+ 2.5\beta_2\beta_3)\sin 18\tau + .2\beta_2\beta_3 \sin 19\tau - (1.1\beta_1\beta_2 + 3.1\beta_2^2$$
$$+ 1.3\beta_2\beta_3)\sin 20\tau - .1\beta_2\beta_3 \sin 21\tau + (.6\beta_1\beta_2 + 2.0\beta_2^2$$

$$+ .7\beta_2\beta_3)\sin 22\tau - (.3\beta_1\beta_2 + 1.1\beta_2^2 + .5\beta_2\beta_3)\sin 24\tau + (.2\beta_1\beta_2 + .7\beta_2^2 + .3\beta_2\beta_3)\sin 26\tau - (.1\beta_1\beta_2 + .4\beta_2^2 + .2\beta_2\beta_3)\sin 28\tau + .2\beta_2^2\sin 30\tau - .1\beta_2^2\sin 32\tau + \cdots.$$

$$x_{2,2}(\tau) = (2770.2\beta_1^2 - 17.0\beta_1\beta_2 - .2\beta_2^2 + 4730.0\beta_1\beta_3 + 27.9\beta_2\beta_3 - 6.9\beta_3^2) - (289.2\beta_1\beta_3 + 757.4\beta_2\beta_3 + 74.9\beta_3^2)\cos\tau + (46102.6\beta_1^2 - 31911.6\beta_1\beta_2 + 17666.4\beta_1\beta_3 + 41787.2\beta_2\beta_3 - 16064.0\beta_3^2)\cos 2\tau + (141.9\beta_1\beta_3 - 29.9\beta_2\beta_3 + 170.2\beta_3^2)\cos 3\tau + (519.3\beta_1^2 - 51.8\beta_1\beta_2 + 4573.9\beta_1\beta_3 + 92.9\beta_2\beta_3 + 992.1\beta_3^2)\cos 4\tau + (35.5\beta_1\beta_3 + 38.6\beta_2\beta_3 + 35.1\beta_3^2)\cos 5\tau + (64.3\beta_1^2 + 36.8\beta_1\beta_2 + 41.8\beta_1\beta_3 + 17.9\beta_2\beta_3 + 10.9\beta_3^2)\cos 6\tau + (4.2\beta_1\beta_3 + 8.4\beta_2\beta_3 + 7.7\beta_3^2)\cos 7\tau - (22.8\beta_1^2 + 26.1\beta_1\beta_2 + 16.7\beta_1\beta_3 - 1.5\beta_2\beta_3 - 1.4\beta_3^2)\cos 8\tau + (.7\beta_1\beta_3 + 2.3\beta_2\beta_3 + 2.4\beta_3^2)\cos 9\tau + (9.5\beta_1^2 + 15.4\beta_1\beta_2 + 9.6\beta_1\beta_3 + 1.2\beta_2\beta_3 + 1.6\beta_3^2)\cos 10\tau + (.4\beta_1\beta_3 + .8\beta_2\beta_3 + .9\beta_3^2)\cos 11\tau - (5.0\beta_1^2 + 8.0\beta_1\beta_2 + 5.0\beta_1\beta_3 - .4\beta_2\beta_3 - .6\beta_3^2)\cos 12\tau + (.2\beta_2\beta_3 + .3\beta_3^2)\cos 13\tau + (2.8\beta_1^2 + 5.3\beta_1\beta_2 + 2.9\beta_1\beta_3 + .2\beta_2\beta_3 + .2\beta_3^2)\cos 14\tau + (.1\beta_2\beta_3 + .1\beta_3^2)\cos 15\tau - (1.4\beta_1^2 + 3.0\beta_1\beta_2 + 1.6\beta_1\beta_3 - .1\beta_3^2)\cos 16\tau + .1\beta_3^2\cos 17\tau + (.8\beta_1^2 + 1.8\beta_1\beta_2 + .9\beta_1\beta_3)\cos 18\tau - (.5\beta_1^2 + 1.0\beta_1\beta_2 + .5\beta_1\beta_3)\cos 20\tau + (.3\beta_1^2 + .8\beta_1\beta_2 + .3\beta_1\beta_3)\cos 22\tau - (.2\beta_1^2 + .6\beta_1\beta_2 + .2\beta_1\beta_3)\cos 24\tau + (.1\beta_1^2 + .4\beta_1\beta_2 + .1\beta_1\beta_3)\cos 26\tau - (.1\beta_1^2 + .2\beta_1\beta_2 + .1\beta_1\beta_3)\cos 28\tau + .1\beta_1\beta_2\cos 30\tau + \cdots.$$

$$w_{2,2}(\tau) = (1229.4\beta_1\beta_3 + 2487.1\beta_2\beta_3 + 327.4\beta_3^2)\sin\tau - (91766.9\beta_1^2 - 63695.3\beta_1\beta_2 + 35062.7\beta_1\beta_3 + 83097.4\beta_2\beta_3 - 21988.4\beta_3^2)\sin 2\tau - (170.3\beta_1\beta_3 - 112.6\beta_2\beta_3 + 254.0\beta_3^2)\sin 3\tau + (1161.5\beta_1^2 + 53.3\beta_1\beta_2 - 4555.3\beta_1\beta_3 - 125.8\beta_2\beta_3 - 1730.8\beta_3^2)\sin 4\tau - (35.4\beta_1\beta_3 + 49.0\beta_2\beta_3 + 52.2\beta_3^2)\sin 5\tau - (84.2\beta_1^2 + 48.0\beta_1\beta_2 + 47.5\beta_1\beta_3 + 21.3\beta_2\beta_3 + 13.5\beta_3^2)\sin 6\tau - (3.7\beta_1\beta_3 + 9.6\beta_2\beta_3 + 9.6\beta_3^2)\sin 7\tau + (26.7\beta_1^2 + 22.4\beta_1\beta_2 + 17.9\beta_1\beta_3 - 3.6\beta_2\beta_3 - 3.6\beta_3^2)\sin 8\tau - (.8\beta_1\beta_3 + 2.6\beta_2\beta_3 + 3.0\beta_3^2)\sin 9\tau - (10.6\beta_1^2 + 13.2\beta_1\beta_2 + 9.8\beta_1\beta_3 + 1.4\beta_2\beta_3 + 1.8\beta_3^2)\sin 10\tau - (.5\beta_1\beta_3 + .8\beta_2\beta_3 + 1.1\beta_3^2)\sin 11\tau + (4.9\beta_1^2 + 7.4\beta_1\beta_3 + 4.7\beta_1\beta_3 - .5\beta_2\beta_3 - .6\beta_3^2)\sin 12\tau - (.3\beta_2\beta_3 + .3\beta_3^2)\sin 13\tau - (2.6\beta_1^2 + 4.5\beta_1\beta_2 + 2.7\beta_1\beta_3 + .2\beta_2\beta_3 + .2\beta_3^2)\sin 14\tau - .1\beta_3^2\sin 15\tau + (1.4\beta_1^2 + 2.7\beta_1\beta_2 + 1.4\beta_1\beta_3 - .1\beta_3^2)\sin 16\tau - (.7\beta_1^2 + 1.6\beta_1\beta_2 + .9\beta_1\beta_3)\sin 18\tau + (.4\beta_1^2 + .9\beta_1\beta_2 + .5\beta_1\beta_3)\sin 20\tau - (.3\beta_1^2 + .6\beta_1\beta_2 + .3\beta_1\beta_3)\sin 22\tau + (.2\beta_1^2 + .4\beta_1\beta_2 + .2\beta_1\beta_3)\sin 24\tau - (.1\beta_1^2 + .3\beta_1\beta_2 + .1\beta_1\beta_3)\sin 26\tau + .1\beta_1\beta_2\sin 28\tau + \cdots.$$

$$x_{3,2}(\tau) = (.3\beta_1^2 - 69.3\beta_1\beta_3 + 5126.0\beta_2^2 - 202.4\beta_2\beta_3 - 10.0\beta_3^2) - (82613.5\beta_1\beta_2 + 7650.0\beta_1\beta_3 + 553.6\beta_2^2 + 36924.0\beta_2\beta_3 - 6499.0\beta_3^2)\cos\tau + (18.2\beta_1\beta_2 - 84.5\beta_1\beta_3 + 1144.6\beta_2^2 - 182.5\beta_2\beta_3 - 25.7\beta_3^2)\cos 2\tau + (.4\beta_1^2 - 28.4\beta_1\beta_2 + 7.7\beta_1\beta_3 - 44.0\beta_2^2 - 67.0\beta_2\beta_3 - 11.8\beta_3^2)\cos 3\tau - (20.3\beta_1\beta_2 + 9.1\beta_1\beta_3 + 7.5\beta_2^2 + 23.8\beta_2\beta_3 + 11.6\beta_3^2)\cos 4\tau - (5.5\beta_1\beta_2 + 4.2\beta_1\beta_3 + 2.7\beta_2^2 + 11.1\beta_2\beta_3 + 4.8\beta_3^2)\cos 5\tau - (.1\beta_1^2 + .5\beta_1\beta_2 + 1.9\beta_1\beta_3 + 1.3\beta_2^2 + 5.2\beta_2\beta_3 + 4.2\beta_3^2)\cos 6\tau - (15.9\beta_1\beta_2 + 1.0\beta_1\beta_3 + .6\beta_2^2 + 3.0\beta_2\beta_3 + 2.3\beta_3^2)\cos 7\tau + (1.0\beta_1\beta_2 - .6\beta_1\beta_3 - .3\beta_2^2 - 1.6\beta_2\beta_3 - 1.3\beta_3^2)\cos 8\tau - (.1\beta_1\beta_2 + .4\beta_1\beta_3 + 1\beta_2^2 + .9\beta_2\beta_3 + .7\beta_3^2)\cos 9\tau - (.7\beta_1\beta_2 + .1\beta_1\beta_3 + .5\beta_2\beta_3 + .1\beta_2^2 + .4\beta_3^2)\cos 10\tau - (.3\beta_1\beta_2 + .1\beta_1\beta_3 + .1\beta_2^2 + .3\beta_2\beta_3 + .3\beta_3^2)\cos 11\tau - (.2\beta_2\beta_3 + .2\beta_3^2)\cos 12\tau + (.2\beta_1\beta_2 - .1\beta_2\beta_3 - .1\beta_3^2)\cos 13\tau + (.1\beta_1\beta_2 - .1\beta_2\beta_3 - .1\beta_3^2)\cos 14\tau - .1\beta_1\beta_2\cos 16\tau + \cdots.$$

$$w_{3,2}(\tau) = (163608.6\beta_1\beta_2 + 15078.4\beta_1\beta_3 + 1357.9\beta_2^2 + 72947.9\beta_2\beta_3 - 12809.7\beta_3^2)\sin\tau + (25.6\beta_1\beta_2 + 83.2\beta_1\beta_3 + 1791.2\beta_2^2 + 204.1\beta_2\beta_3 + 25.3\beta_3^2)\sin 2\tau + (.1\beta_1^2 + 57.4\beta_1\beta_2 - 5.1\beta_1\beta_3 + 82.3\beta_2^2 + 64.4\beta_2\beta_3 + 7.8\beta_3^2)\sin 3\tau - (6.7\beta_1\beta_2 - 4.5\beta_1\beta_3 - 24.2\beta_2^2 + 18.0\beta_2\beta_3 + 5.7\beta_3^2)\sin 4\tau + (8.2\beta_1\beta_2 + 1.7\beta_1\beta_3 + 9.1\beta_2^2 + 8.6\beta_2\beta_3 + 1.9\beta_3^2)\sin 5\tau + (1.8\beta_1\beta_2 + .7\beta_1\beta_3 + 4.9\beta_2^2 + 4.1\beta_2\beta_3 + 1.4\beta_3^2)\sin 6\tau - (.8\beta_1\beta_2 - .3\beta_1\beta_3 - 2.4\beta_2^2 - 2.7\beta_2\beta_3 - .7\beta_3^2)\sin 7\tau - (.5\beta_1\beta_2 - .1\beta_1\beta_3 - 1.4\beta_2^2 - 1.5\beta_2\beta_3 - .3\beta_3^2)\sin 8\tau + (.3\beta_1\beta_2 + .1\beta_1\beta_3 + .7\beta_2^2 + .9\beta_2\beta_3 + .2\beta_3^2)\sin 9\tau + (.6\beta_1\beta_2 + .4\beta_2^2 + .6\beta_2\beta_3 + .1\beta_3^2)\sin 10\tau + (.3\beta_1\beta_2 + .3\beta_2^2 + .3\beta_3\beta_2 + .1\beta_3^2)\sin 11\tau + (.1\beta_1\beta_2 + .2\beta_2^2 + .2\beta_2\beta_3)\sin 12\tau - (.1\beta_1\beta_2 - .1\beta_2^2 - .1\beta_2\beta_3)\sin 13\tau + (.1\beta_1\beta_2 + .1\beta_2^2 + .1\beta_2\beta_3)\sin 14\tau + .1\beta_2\beta_3\sin 15\tau + .1\beta_1\beta_2\sin 16\tau + \cdots.$$

The above coefficients, when multiplied by the proper powers of μ, give the values of the coördinates, so far as they depend upon the terms of the first and second orders, accurate to nine decimal places. Which places would be affected by terms of higher orders cannot be definitely stated without information as to (1) whether the series (40) converge for $\mu = .0001$, and (2) how rapidly they converge. Although the question of convergence for this value has not yet been answered, it is not improbable that the radius of convergence, whose existence has been proved above, is greater than .0001.

FURTHER NOTE ON MACLAURIN'S SPHEROID*

BY

GEORGE HOWARD DARWIN

In volume 4 (1903) of these Transactions a paper of mine was published on the approximate determination of Maclaurin's ellipsoid. It was there shown that it is possible to determine the form of the ellipsoid and its angular velocity by means of spherical harmonic analysis to a higher order of approximation than is usually supposed to be attainable by that method.

It appears that we ought to be able by means of the considerations adduced to determine the angular velocity corresponding to a given ellipticity as far as the cube of that ellipticity, and I was much puzzled to explain why I failed to obtain correctly the term involving the cube.

On page 117 of the paper I gave reasons for believing that the determination of this last term would prove to be illusory. Having had occasion recently to examine carefully another piece of work I saw that the reason given was in its turn fallacious, and that this last term should be attainable. Accordingly I looked carefully over the analysis to see whether there were any terms missing in my calculation, and having found the source of the discrepancy I propose in this note to supply the defect. No attempt will be made to make this new investigation complete, so that the present paper will only be intelligible in conjunction with the previous one.

The equation to the spheroid which is to be made a figure of equilibrium under rotation ω was written

$$\tau = -eS_2 - fS_4 - \Sigma f_i' S_i',$$

where $\tau = (a^3 - r^3)/3a^3$, $S_2 = \frac{1}{3} - \mu^2$, and S_4 and S_i' are the spherical surface harmonics of μ the cosine of colatitude from the axis of rotation.

On page 118 I showed that the potential of the sphere from which the ellipsoid is derived is

$$V = \tfrac{2}{3}\pi\rho(3a^2 - r^2),$$

and I put $r^2 = a^2(1 - 2\tau - \tau^2)$. Now one more term in the development of r^2 should have been included, and I ought to have written

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$$r^2 = a^2(1 - 2\tau - \tau^2 - \tfrac{4}{3}\tau^3 \cdots).$$

In order to evaluate the additional term in the part of the energy denoted $-SR$ another integral is required, besides those given on page 117.

It is, say,

$$\sigma_4 = \frac{3}{4\pi}\int (S_2)^4 d\sigma .$$

By the results given on page 117 for $(S_2)^2$ we have

$$\sigma_4 = \frac{3}{4\pi}\int \left[\frac{\omega_0}{\phi_0}S_0 + \frac{\omega_2}{\phi_2}S_2 + \frac{\omega_4}{\phi_4}S_4\right]^2 d\sigma$$

$$= \frac{(\omega_0)^2}{\phi_0} + \frac{(\omega_2)^2}{\phi_2} + \frac{(\omega_4)^2}{\phi_4}.$$

With the numerical values given on the same page, I find

$$\sigma_4 = \frac{16}{3^2.5.7}.$$

Returning to page 118 it is easy to see that the additional term in SR is $(M^2/a)\frac{1}{8}e^4\sigma_4$.

Hence the additional term in equation (5) which gives the value of $\frac{1}{2}SS - SR$ is $(M^2/a)(-\frac{1}{8}e^4\sigma_4)$.

On page 119 I avowedly dropped a term in CR, viz.:

$$-\frac{9M^2}{4\pi a}\int \frac{1}{3.5}e^4(S_2)^4 d\sigma = \frac{M^2}{a}\left[-\frac{1}{5}e^4\sigma_4\right].$$

Next on page 127 I avowedly dropped a term in the energy $\frac{1}{2}DD$, viz.:

$$\frac{1}{2}\frac{M\rho}{a}\int a^3 e^4 (S_2)^4 d\sigma = \frac{M^2}{a}\left[\frac{1}{2}e^4\sigma_4\right].$$

Lastly on page 129 I omitted the whole of the energy given by the formula (12). It is

$$\frac{M^2}{a}\left[-\frac{9}{20}\frac{(\omega_2)^2}{\phi_2} - \frac{5}{6}\frac{(\omega_4)^2}{\phi_4}\right]e^4.$$

Thus all the omitted portions together are

$$\frac{M^2}{a}\left[\frac{2}{3.5}\sigma_4 - \frac{9}{20}\frac{(\omega_2)^2}{\phi_2} - \frac{5}{6}\frac{(\omega_4)^2}{\phi_4}\right]e^4.$$

With the numerical values given above the additional term in the lost energy

of the system (omitted because I thought terms in e^4 would lead to no further accuracy) is found to be

$$\frac{M^2}{a}\left[-\frac{16}{3^3.7^2}e^4\right].$$

In calculating the moment of inertia C_r on page 129, I omitted purposely

$$-\frac{3M^2}{4\pi a}\cdot\frac{3}{4\pi\rho}\int[\tfrac{2}{3}+S_2][-\tfrac{1}{3}e^3(S_2)^2]d\sigma^* = \frac{3M^2e^3}{4\pi\rho a}(\tfrac{2}{9}\omega_2+\tfrac{1}{3}\sigma_4).$$

Hence this with its sign changed and multiplied by $\frac{1}{2}\omega^2$ is an additional term in (14) giving $\frac{1}{2}C\omega^2$. Thus (14) should run

$$\tfrac{1}{2}C\omega^2 = \frac{M^2}{a}\frac{\omega^2}{4\pi\rho}[\tfrac{3}{5}+\tfrac{3}{2}e\phi_2+e^2(\phi_2+\tfrac{3}{2}\omega_2)-e^3(\tfrac{1}{3}\omega_2+\tfrac{1}{2}\sigma_4)+3ef\omega_4].$$

It is convenient to insert numerical values throughout, instead of retaining symbols as I did before.

I find then that the first formula of § 7 will run

$$E\div\frac{M^2}{a} = -\frac{4}{3.5^2}e^2-\frac{64}{3^3.5^2.7}e^3-\frac{16}{3^3.7^2}e^4+\frac{104}{3^2.5^2.7}e^2f-\frac{1}{3^2}f^2-\Sigma\frac{i-1}{2i+1}(f_i')^2\phi_i'$$
$$+\frac{\omega^2}{4\pi\rho}\left[\frac{3}{5}+\frac{2}{5}e+\frac{4}{3.7}e^2-\frac{8}{3^3.5.7}e^3+\frac{8}{5.7}ef\right].$$

In this the terms in e^4 in the first part and in e^3 in the second part are the additional terms now included.

Since $\partial E/\partial f_i' = 0$ is one of the conditions for the figure of equilibrium, it follows at once that $f_i' = 0$, for all values of i and s.

By neglecting all terms above those of the second order, and putting $\partial E/\partial e = 0$, we see that as a first approximation $\omega^2/4\pi\rho = 4e/3.5$ On putting $\partial E/\partial f = 0$ we obtain

$$\frac{104}{3^2.5^2.7}e^2-\frac{2}{3^2}f+\frac{\omega^2}{4\pi\rho}\frac{8}{5.7}e = 0.$$

Then by means of the first approximation for $\omega^2/4\pi\rho$, we find

$$f = \frac{4}{7}e^2.$$

On substituting this value for f in the expression for the energy, we have

$$E\div\frac{M^2}{a} = -\frac{4}{3.5^2}e^2-\frac{64}{3^3.5^2.7}e^3-\frac{352}{3^3.5^2.7^2}e^4+\frac{\omega^2}{4\pi\rho}\left[\frac{3}{5}+\frac{2}{5}e+\frac{4}{3.7}e^2+\frac{808}{3^3.5.7^2}e^3\right].$$

* In line 7 from foot of page 129 there is a misprint; the double integral should obviously be a single one, and $d\tau$ should be deleted.

On equating $\partial E/\partial e$ to zero we find

$$\frac{4}{3.5}e\left(1+\frac{8}{3.7}e+\frac{176}{3^2.7^2}e^2\right)=\frac{\omega^2}{4\pi\rho}\left[1+\frac{20}{3.7}e+\frac{404}{3^2.7^2}e^2\right].$$

Whence

$$\frac{\omega^2}{4\pi\rho}=\frac{4}{3.5}e\left(1-\frac{4}{7}e+\frac{4}{3.7^2}e^2\right).$$

Only the first two terms were evaluated in the former paper, and this is the required new result.

The equation to the ellipsoid, as calculated in this way, is

$$\frac{r^3}{a^3}=1+3(eS_2+\tfrac{4}{7}e^2S_4).$$

The method then will give the coefficient of S_4 only as far as e^2, and fails to give the coefficient of S_6. Nevertheless it gives the square of the angular velocity as far as e^3.

It remains to verify the correctness of our result.

It may be proved, as in the former paper, that the equation to an ellipsoid whose equatorial and polar semi-axes are a_1 and $a_1(1-e_1)$ may be written in the form

$$r^3=a_1^3(1-e_1)[1+3(e_1+\tfrac{5}{14}e_1^2+\tfrac{2}{21}e_1^3)S_2+\tfrac{12}{7}e_1^2(1+\tfrac{9}{11}e_1)S_4-\tfrac{40}{33}e_1^3S_6].$$

Now a denotes the mean radius of our spheroid, so that

$$a^3=a_1^3(1-e_1).$$

Since $3e$, $3f$, and say $3g$ are to be the coefficients S_2, S_4, S_6, in the equation to the ellipsoid

$$r^3=a^3[1+3eS_2+3fS_4+3gS_6],$$

we have

$$e=e_1+\tfrac{5}{14}e_1^2+\tfrac{2}{21}e_1^3,$$

$$f=\tfrac{4}{7}e_1(1+\tfrac{9}{11}e_1),$$

$$g=-\tfrac{40}{99}e_1^3.$$

By inverting the expression for e, we find

$$e_1=e-\frac{5}{14}e^2+\frac{47}{2.3.7^2}e^3,$$

$$f=\frac{4}{7}e^2\left(1+\frac{8}{77}e\right),$$

$$g=-\frac{40}{99}e^3.$$

If the eccentricity of the ellipsoid be denoted by $\sin\gamma$, we have

$$\cos\gamma = 1 - e_1, \qquad \sin^2\gamma = 2e_1\left(1 - \frac{1}{2}e_1\right).$$

Hence

$$\cos\gamma\sin^2\gamma = 2e_1\left(1 - \frac{3}{2}e_1 + \frac{1}{2}e_1^2\right) = 2e\left(1 - \frac{13}{7}e + \frac{509}{2.3.7^2}e^2\right),$$

$$\cos\gamma\sin^4\gamma = 4e_{\frac{1}{2}}^2(1 - 2e_1) = 4e^2\left(1 - \frac{19}{7}e\right),$$

$$\cos\gamma\sin^6\gamma = 8e_1^3 = 8e^3.$$

Now the rigorous solution for the angular velocity of Maclaurin's ellipsoid is

$$\frac{\omega^2}{4\pi\rho} = \cos\gamma\sum_1^\infty \frac{(2n-1)!\sin^{2n}\gamma}{(2n+1)(2n+3)[(n-1)!]^2 2^{2n-3}}$$

$$= \frac{2}{3.5}\cos\gamma\sin^2\gamma + \frac{3}{5.7}\cos\gamma\sin^4\gamma + \frac{5}{2^2.3.7}\cos\gamma\sin^6\gamma\cdots.$$

Substituting for γ in terms of e, we find

$$\frac{\omega^2}{4\pi\rho} = \frac{4}{3.5}e\left(1 - \frac{4}{7}e + \frac{4}{3.7^2}e^2\right).$$

This agrees, as it should, with the result found by means of spherical harmonic analysis.

I do not think that the additional term found suffices to assign a limit to the stability of the ellipsoid, as I conjectured that it would do.

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POTENTIAL FUNCTIONS ON THE BOUNDARY OF THEIR REGIONS OF DEFINITION*

BY

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In a memoir on PICARD's method of successive approximations,† DINI points out the necessity of knowing that the potential functions used have first derivatives which are bounded even in the neighborhood of points of the boundary of their region of definition. These functions are determined by their boundary values, and for the case of the circle DINI finds the following condition on the boundary values $f(\vartheta)$ sufficient to insure limited derivatives for the potential function approaching them: $|f'(\vartheta)|$ shall be finite and integrable, and the integral

$$\int \left| \frac{f'(\vartheta+t)-f'(\vartheta-t)}{t} \right| dt$$

shall be convergent when extended over an interval containing $t=0$ in its interior, for every value of ϑ.‡ Here ϑ is understood to be the angle subtended at the center by an arc of the circle between a fixed and a variable point. He then remarks that the theorem holds for other regions which can be conformally represented upon the circle by transformations which bring no singularities with them, clearly not intending this to be a complete answer to the problem, for the existence of the transformations mentioned is a special case of the general question of the behavior of the derivatives of a potential function defined by its boundary values.§ Aside from its importance to the method of successive approximations, an investigation of the behavior of the derivatives of a potential

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† Acta Mathematica, vol. 25 (1902), p. 185.

‡ It seems worth noticing that the continuity of $f'(\vartheta)$ is insufficient. For if $f'(\vartheta)$ be defined in the neighborhood of $\vartheta=0$ as $\vartheta/|\vartheta|\sqrt{\log 1/|\vartheta|}$ for $\vartheta \neq 0$ and as $f'(0)=0$ for $\vartheta=0$, and be elsewhere arranged to be continuous and periodic, the potential function approaching the boundary values $f(\vartheta)$ will have a derivative in the direction of the radius which becomes infinite in the neighborhood of the point $\vartheta=0$. On the other hand DINI's condition does not imply the continuity of $f'(\vartheta)$.

§ The mapping is dependent upon the Green's function $\log 1/\rho + v$, where v is a potential function taking on the same boundary values as $-\log 1/\rho$. But $\log 1/\rho$ is analytic, so that the character of the derivatives of the mapping transformations is essentially that of the derivatives of v.

function in the neighborhood of boundary points finds application in the theory of functions of a complex variable and in questions of mathematical physics,* notably such as make use of Green's functions.

The first treatment of this topic appears to be due to CAPELLI;† his work for the general region depends upon a hypothesis regarding the behavior of a second derivative of the potential function. For the case of the circle he shows that the first and second derivatives of the potential function approaching the boundary values $f(\vartheta)$ are limited provided $f'(\vartheta)$ exists and $[f'(\vartheta+t)-f'(\vartheta)]/t$ is limited. For the continuity of the first derivatives within and upon the boundary C. NEUMANN‡ finds the conditions that the boundary curve have continuous curvature, and that the boundary values have a continuous first derivative and a second derivative which is continuous except at isolated points. PAINLEVÉ§ considers functions $X+iY$ of a complex variable $x+iy$ in a region bounded by a curve $x=x(s)$, $y=y(s)$, s being the length of arc. If $x(s)$ and $y(s)$ have $p+2$ derivatives and if the function $X(s)$ which X approaches on the boundary has an integrable $(p+1)$th derivative, $X+iY$ has limited derivatives of pth and lesser orders. The conditions found by KORN‖ for the finiteness of the derivatives of potential functions of a three-dimensional region involve the existence of other functions taking on the required boundary values and satisfying certain continuity requirements on their derivatives of first and second orders. In all cases the results admit of the generalization contained in the following pages.

§ 1. *Definitions. Statement of hypotheses and main conclusions.*

In the present paper we shall deal with a simply connected closed region R of the x, y-plane bounded by a curve C, the coördinates of whose points we shall think of as given by functions $x(s)$, $y(s)$ of the length of arc s measured from a fixed point. Points "within" R means points of R not belonging to C. A "potential function" of R means a function u of x and y which together with its first derivatives is continuous, and which has second derivatives satisfying the relation $\Delta u=\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$ at all points *within* R. We shall be concerned with potential functions which approach boundary values $f(s)$ as (x,y) approaches in any manner in R a boundary point $[x(s),y(s)]$; the boundary

*See the important memoir of LIAPOUNOFF, *Sur certaines questions qui se rattachent au problème de Dirichlet*, Journal de Mathematiques, ser. 5, vol. 4 (1898), p. 241.

†*Sopra l'integrale dell' equazione alle derivate potenziali di Laplace*, Giornale di matematiche di Battaglini, vol. 13 (1885), pp. 123–157.

‡*Über die Methode des arithmetischen Mittels, Zweite Abhandlung*, Leipziger Abhandlungen, vol. 14 (1888), pp. 562–726.

§*Sur la theorie de la répresentation conforme*, Comptes Rendus, vol. 112 (1891), pp. 653–657.

‖*Über die Lösungen des Dirichlet'schen Problems welche durch eine Combination der Methoden von Neumann und Schwarz gefunden werden*, Mathematische Annalen, vol. 53 (1900), pp. 593–608.

values $f(s)$ we shall assume continuous. Such conditions are known to determine u if it exists; and R is said to be the "region of definition" of u. We shall seek *conditions upon* $x(s)$, $y(s)$ *and upon* $f(s)$ *which suffice that the derivatives* $\partial u/\partial x$ *and* $\partial u/\partial y$ *approach finite values at each point* $[x(s), y(s)]$ *of the boundary, these values being independent of the manner of approach in* R. From the conditions found it will also follow that these derivatives are bounded.

Previous investigations of this character have assumed a boundary curve with finite, if not continuous curvature at each point. Consideration of the conformal mapping problem and the Green's function, however, make it appear important to lessen the assumptions with respect to C, and, if necessary, to increase those upon $f(s)$. For the derivatives of the Green's function in the neighborhood of the boundary will evidently have the same continuity characteristics as those of of the additive potential function v which takes on the same boundary values as $\log \rho$, that is, $f(s) = \frac{1}{2} \log [(\xi - x(s))^2 + (\eta - y(s))^2]$. But these values have the same continuity characteristics as $x(s)$, $y(s)$. We are thus led to seek *conditions which bear, as nearly as possible, equally upon boundary curve and upon boundary values*, a consideration which seems thus far to have escaped attention.

With this in mind one would be led to attempt to obtain DINI's results for regions bounded by curves whose $x(s)$ and $y(s)$ satisfy simply his conditions upon $f(\vartheta)$. That the results remain true is probable, but I have not been able to attain quite this degree of generality. The results I have succeeded in establishing may be stated as follows.

Let R be a simply connected closed region of the x, y-plane bounded by a curve C with parametric equations $x = x(s)$, $y = y(s)$, where s denotes the length of arc of C measured from a fixed point, and let $f(s)$ be a function of the position of a point on C. We then have the following theorem.

(*A*) *If three positive numbers* N, α *and* δ *independent of* s *and* Δs *can be found such that*

$$|x'(s+\Delta s) - x'(s)| < N|\Delta s|^\alpha, \qquad |y'(s+\Delta s) - y'(s)| < N|\Delta s|^\alpha$$

as soon as $|\Delta s| < \delta$;* and

* Conditions of this sort are fairly frequent in potential theory. If we omit the demand that N, α and δ be independent of s, the functions $x'(s)$, $y'(s)$ would be termed by KORN (*Lehrbuch der Potentialtheorie*, vol. 2, p. vi) "regular," and the curve C would be a "regular curve." As the word "regular" has been used in a different sense (see *Encyklopädie der Mathematischen Wissenschaften*, vol. 2, B. 1, p. 9, footnote) I should propose the more connotative expression, "algebraically continuous," suggested by the behavior of algebraic functions at branch-points where they are finite. The functions $x'(s)$ and $y'(s)$ of condition (*A*) have algebraic continuity which is uniform, and the constant α we shall call the "*index*" of algebraic continuity. It is interesting to note that algebraic continuity in a closed interval does not imply uniform algebraic continuity.

(B) if $f'(s)$ is continuous and such that the integral

$$\int_0^\tau \left| \frac{f'(s+t) - f'(s-t)}{t} \right| dt$$

*is convergent and vanishes with τ uniformly with respect to s;**

then there exists a uniquely determined potential function of R, $u(x, y)$, which approaches the boundary values $f(s)$; it has first derivatives which approach finite limits on the boundary and these limits form each a continuous function of s; if $u(x, y)$ be defined at the boundary by the values it approaches, the (in general one-sided) derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist at the boundary and coincide with the limits of these derivatives formed for interior points, so that $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in the closed region R.†

The proof of this theorem depends upon the representation of a potential function as the potential of a double distribution on C. In a study of this question in a second paper by means of the method given by FREDHOLM,‡ I shall show that $u(x, y)$ exists and is equal in the interior of R to the potential $W(x, y)$ of a double distribution on C with a moment $\phi(s)$ satisfying condition (B). In this paper we shall limit ourselves to a study of the double distribution $W(x, y)$, and find that if its moment $\phi(s)$ satisfies condition (B), the potential function $u(x, y)$ coinciding with $W(x, y)$ within R and defined at the boundary so as to be continuous has the properties enunciated above. This will establish the theorem.

§ 2. *The normal derivatives of the potential of a double distribution.*

Let the potential of the double distribution be given in the form

$$W(\xi, \eta) = \int_0^l \phi(t) \frac{\partial}{\partial t} \arctan \frac{y(t) - \eta}{x(t) - \xi} dt = 2\pi\phi(0) - \int_0^l \phi'(t) \arctan \frac{y(t) - \eta}{x(t) - \xi} dt$$

or

$$W(\xi, \eta) = -\int_0^l \phi'(t) \arctan \frac{y(t) - \eta}{x(t) - \xi} dt,$$

* Two special cases may be mentioned in which condition (B) may be lightened. First, if R is the surface of a circle, the theorem holds if $f'(s)$ is continuous, and if the integral

$$\int_0^\tau \frac{f'(s+t) - f'(s-t)}{t} dt$$

vanishes uniformly with τ. Secondly, for the general region where C satisfies condition (A) if we drop the demand for uniformity in (B), we may still assert that the derivatives $\partial u/\partial x$ and $\partial u/\partial y$ approach finite limits as the point (x, y) approaches a boundary point $[x(s), y(s)]$ *along the normal* through it, and these limits form each a continuous function of s. We cannot however argue either uniformity of the approach or that the limit is independent of the direction of the approach.

† By "approach" in the above theorem is meant the general approach along any infinite set of points in R.

‡ *Oefversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, 1900.

according as (ξ, η) lies within or without C; l is the length of C. We first consider the derivative in the direction of a normal ν to C through (ξ, η). It is given by

$$\frac{\partial W}{\partial \nu} = -\int_0^l \phi'(t) \frac{\partial}{\partial \nu} \arctan \frac{y(t)-\eta}{x(t)-\xi} dt = \int_0^l \phi'(t) \frac{\partial}{\partial \sigma} \log \frac{1}{\rho} dt,$$

where

$$\rho^2 = [\xi - x(t)]^2 + [\eta - y(t)]^2$$

and where σ denotes the direction perpendicular to ν and related to it as the positive x-axis to the positive y-axis. We shall prove in this section that as (x, y) approaches $[x(s), y(s)]$ along the normal, $\partial W/\partial \nu$ approaches a finite limit; that this limit is the same whether the approach is from within or without C, and that the approach is in both cases uniform with respect to s. To do this we write

$$\frac{\partial W}{\partial \nu} = J_1 + J_2$$

where

$$J_1 = \int_0^{s-\tau} \phi'(t) \frac{\partial}{\partial \sigma} \log \frac{1}{\rho} dt + \int_{s+\tau}^{l} \phi'(t) \frac{\partial}{\partial \sigma} \log \frac{1}{\rho} dt$$

and

$$J_2 = \int_{s-\tau}^{s+\tau} \phi'(t) \frac{\partial}{\partial \sigma} \log \frac{1}{\rho} dt,$$

in which we impose upon τ a first restriction that a circle with center at $[x(s), y(s)]$ through the nearer of the two points $[x(s+\tau), y(s+\tau)]$ and $[x(s-\tau), y(s-\tau)]$ shall cut from C only one arc, namely that passing through $[x(s), y(s)]$. As C has a continuously turning tangent and no double points, one value for this upper limit for τ may be found which will serve for all values of s, as may be easily verified. Now for fixed τ the distance

$$\rho(s, s+\tau) = \sqrt{[x(s+\tau) - x(s)]^2 + [y(s+\tau) - y(s)]^2}$$

has a positive minimum; call it r. Then within the circle

$$[\xi - x(s)]^2 + [\eta - y(s)]^2 = (r/2)^2$$

J_1 is a continuous function* of (ξ, η), a point on the normal to C through $[x(s), y(s)]$, and the continuity is uniform with respect to ξ, η and s. On the other hand it will be shown that τ may be taken so small that independently of s the inequality $|J_2| < \epsilon/4$ holds for any point (ξ, η) of the normal, except $[x(s), y(s)]$, ϵ being an arbitrary positive constant. Determining τ so as to meet both the above demands, and then r, we can, because of the continuity of

*In fact its first derivative is continuous and has a finite upper limit independent of s, being the second derivative of a potential within a region included within a region containing no masses.

J_1, determine $\kappa < r/2$ such that for any two points (ξ, η) and (ξ', η') of the normal within the circle $[\xi - x(s)]^2 + [\eta - y(s)]^2 = \kappa^2$ we have

$$|J_1(\xi', \eta') - J_1(\xi, \eta)| < \frac{\epsilon}{2}$$

independently of s. Then also since

$$|J_2(\xi', \eta') - J_2(\xi, \eta)| \leqq |J_2(\xi', \eta')| + |J_2(\xi, \eta)| < \frac{\epsilon}{2},$$

we have

$$\left| \frac{\partial}{\partial \nu} W(\xi', \eta') - \frac{\partial}{\partial \nu} W(\xi, \eta) \right| < \epsilon.$$

Hence, *$\partial W(\xi, \eta)/\partial \nu$ approaches a finite limit as (ξ, η) approaches $[x(s), y(s)]$ along the normal to C through this point, the approach of $\partial W(\xi, \eta)/\partial \nu$ being uniform with respect to s, and the limit being the same whether the approach of (ξ, η) be from within or from without C.**

To prove the stated property of J_2, let us take for axes the tangent and normal through $[x(s), y(s)]$, and for simplicity in writing let us measure s from this point as origin. Then the integral

$$J_2 = \int_{-\tau}^{+\tau} \phi'(t) \frac{\partial}{\partial x} \log \frac{1}{\rho}\, dt = \int_0^{\tau} \left\{ \phi'(t) \frac{\partial}{\partial x} \log \frac{1}{\rho(t)} + \phi'(-t) \frac{\partial}{\partial x} \log \frac{1}{\rho(-t)} \right\} dt$$

may be written in the form $J_{21} + J_{22}$ where

$$J_{21} = \int_0^{\tau} [\phi'(t) - \phi'(-t)] \frac{\partial}{\partial x} \log \frac{1}{\rho}\, dt$$

and

$$J_{22} = \int_0^{\tau} \phi'(-t) \left\{ \frac{\partial}{\partial x} \log \frac{1}{\rho(t)} + \frac{\partial}{\partial x} \log \frac{1}{\rho(-t)} \right\} dt.$$

For the first of these two integrals we find an upper limit as follows:

$$|J_{21}| \leqq \int_0^{\tau} |\phi'(t) - \phi'(-t)| \cdot \left| \frac{\partial}{\partial x} \log \frac{1}{\rho} \right| dt = \int_0^{\tau} |\phi'(t) - \phi'(-t)| \cdot \left| \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right| dt.$$

In this we have

$$\left| \frac{\partial \rho}{\partial x} \right| = |\cos(\rho, x)| \leqq 1,$$

and ρ is not less than the absolute value of its projection $x(t)$. But the latter satisfies the condition

$$|x(t)| = \frac{\left| \int_0^x \sqrt{2}\, dx \right|}{\sqrt{2}} \geqq \frac{\left| \int_0^x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \right|}{\sqrt{2}} = \frac{|t|}{\sqrt{2}}$$

*Compare the theorem of LIAPOUNOFF, loc. cit., p. 299.

if we so restrict τ that

$$\left|\frac{dy}{dx}\right| = \left|\frac{y'(t)}{x'(t)}\right| < 1,$$

and this is possible because $y'(0) = 0$, $x'(0) = 1$ and $x'(t)$ and $y'(t)$ are uniformly continuous. Moreover, the restriction can be made so as to hold equally for all values of s. Hence

$$|J_{21}| \leqq \sqrt{2} \int_0^\tau \left|\frac{\phi'(t) - \phi'(t)}{t}\right| dt,$$

which by hypothesis (B) can be made less than $\epsilon/8$ by restricting τ sufficiently, the restriction holding uniformly with respect to s.

Turning to J_{22}, as (ξ, η) is on the normal, we have $\xi = 0$, and hence

$$J_{22} = \int_0^\tau \phi'(-t) \left\{ \frac{x(t)}{\rho^2(t)} + \frac{x(-t)}{\rho^2(-t)} \right\} dt,$$

so that

$$|J_{22}| \leqq \max |\phi'(t)| \int_0^\tau \left| \frac{x(t)}{\rho^2(t)} + \frac{x(-t)}{\rho^2(-t)} \right| dt.$$

The integrand is a continuous function of τ and η except at the point $(0, 0)$. We shall show it to be always less in absolute value than a finite constant multiplied by $t^{\alpha-1}$ where α is the index of algebraic continuity of condition (A). Hence, as $\phi'(t)$ is continuous, $|J_{22}|$ can be made less than $\epsilon/8$ by restricting τ, independently of the value of s, and hence $|J_2| \leqq |J_{21}| + |J_{22}| < \epsilon/4$, as we were to show.

To prove the property stated for the integrand of J_{22}, note that because $x(0) = y(0) = y'(0) = 0$ and $x'(0) = 1$, we have by applying the law of the mean and using hypothesis (A):

$$\begin{aligned}
x(t) &= x'(\vartheta t) \cdot t = t(1 + F \cdot t^\alpha), & y(t) &= F \cdot t^{1+\alpha}, \\
x(-t) &= -t(1 + F \cdot t^\alpha), & y(-t) &= F \cdot t^{1+\alpha}, \\
\rho^2(t) &= t^2(1 + F \cdot t^\alpha)^2 + (F \cdot t^{1+\alpha} - \eta)^2, \\
\rho^2(-t) &= t^2(1 + F \cdot t^\alpha)^2 + (F \cdot t^{1+\alpha} + \eta)^2,
\end{aligned} \tag{1}$$

where each F denotes a finite function and where we have supposed $t(\leqq \tau) < \delta$ in order that hypothesis (A) might be in force. If now we write

$$N(\eta, t) = x(t)\rho^2(-t) + x(-t)\rho^2(t)$$

and

$$D(\eta, t) = \rho^2(t)\rho^2(-t),$$

the integrand in $|J_{22}|$ is $N(\eta, t)/D(\eta, t)$. Employing the equations (1) we find

$$|N(\eta, t)| < Gt^{1+\alpha}(t^2 + t|\eta| + \eta^2),$$

where G is a finite constant, and

$$D(\eta, t)=\rho^2(t)\rho^2(-t)=(t^2+\eta^2+F\cdot t^{2+a}-F\cdot t^{1+a}\cdot\eta)(t^2+\eta^2+F\cdot t^{2+a}+F\cdot t^{1+a}\cdot\eta)$$
$$>[t^2+\eta^2-Gt^a(t+t|\eta|)]^2.$$

If now τ be further so restricted that $Gt^a<\frac{1}{2}$, that is, $\tau<1/(2G)^{1/a}$, then

$$\left|\frac{N(\eta, t)}{D(\eta, t)}\right|<\frac{4Gt^{a-1}\left(1+\frac{|\eta|}{t}+\frac{\eta^2}{t^2}\right)}{\left(1-\frac{|\eta|}{t}+\frac{2\eta^2}{t^2}\right)^2}\leqq 4GMt^{a-1},$$

where M is the maximum of the bounded function $(1+x+x^2)/(1-x+2x^2)^2$. This gives us the desired inequality. Thus the result enunciated with respect to the normal derivatives is fully justified.*

§ 3. *The tangential derivatives of the potential of a double distribution.*

A similar result may be obtained for the derivatives of $W(\xi, \eta)$ in a direction σ perpendicular to the normal. Using the same coördinate system as before, we have

$$\frac{\partial}{\partial\sigma}W(\xi, \eta)=\frac{\partial}{\partial\xi}W(\xi, \eta)=-\int_0^l\phi'(t)\frac{\partial}{\partial\xi}\arctan\frac{\eta-y(t)}{\xi-y(t)}dt$$
$$=\int_0^l\phi'(t)\frac{\eta-y(t)}{\rho^2}dt=J_1+J_2,$$

where

$$J_1=\int_{-l/2}^{-\tau}\phi'(t)\frac{\eta-y(t)}{\rho^2}dt+\int_\tau^{l/2}\phi'(t)\frac{\eta-y(t)}{\rho^2}dt \text{ and } J_2=\int_{-\tau}^{\tau}\phi'(t)\frac{\eta-y(t)}{\rho^2}dt.$$

Again J_1 is continuous for fixed τ, and uniformly so in ξ, η and s. If $J_2=J_{21}+J_{22}$, where

$$J_{21}=\int_0^\tau[\phi'(t)-\phi'(-t)]\frac{\eta-y(t)}{\rho^2}dt$$

and

$$J_{22}=\int_0^\tau\phi'(-t)\left\{\frac{\eta-y(t)}{\rho^2(t)}+\frac{\eta-y(-t)}{\rho^2(-t)}\right\}dt,$$

then J_{21} again vanishes with τ independently of η and s. However, J_{22} de-

* To obtain an expression for the limit of the normal derivatives of $W(\xi, \eta)$, which we denote by $W_n(s)$, a slight extension of the above work leads us to the result

$$W_n(s)=\int_0^s\phi'(t)\frac{\partial}{\partial s}\log\frac{1}{\bar\rho}dt$$

where $\bar\rho^2=[x(t)-x(s)]^2+[y(t)-y(s)]^2$ and where the Cauchy principal value of this integral is to be taken. This may be proven continuous by a direct examination. We shall have another proof of the continuity of $W_n(s)$ later.

mands a different treatment, for it does not vanish with τ. We put $J_{22} = J_{221} + J_{222}$, where

$$J_{221} = 2\int_0^{\tau} \phi'(-t)\frac{\eta}{\eta^2 + t^2}dt$$

and

$$J_{222} = \int_{-\tau}^{+\tau} \phi'(-t)\left\{\frac{\eta - y(t)}{\rho^2(t)} - \frac{\eta}{\eta^2 + t^2}\right\}dt.$$

Here $J_{221} = 2\phi'(-\bar{t})$ arc tan τ/η where $0 \leqq \bar{t} \leqq \tau$ and $0 \leqq$ arc tan $\tau/\eta \leqq \pi/2$. To fix ideas, let us suppose we are approaching the boundary from within. It is legitimate to restrict η to be less than τ^2. Then

$$\text{arc tan}\frac{1}{\sqrt{\eta}} < \text{arc tan}\frac{\tau}{\eta} < \frac{\pi}{2};$$

thus by restricting first τ and then η we can make $\phi'(-\bar{t})$ as close to $\phi'(0)$ and arc tan τ/η as close to $\pi/2$ as we please, and accordingly we may make

$$|J_{221} - \pi\phi'(0)| < \frac{\epsilon}{12}.$$

Hence for any two points $(0, \eta)$ and $(0, \eta')$ so restricted we have

$$|J_{221}(0, \eta') - J_{222}(0, \eta)| < \epsilon/6. \tag{2}$$

For J_{222} we find by use of equations (1),

$$|J_{222}| \leqq \max|\phi'(t)| \cdot \int_{-\tau}^{+\tau} \left|\frac{y(t)\{\eta^2 - \eta y(t) - t^2\} + \eta\{t^2 - x^2(t)\}}{\rho^2(t)\cdot\{\eta^2 + t^2\}}\right| dt$$

$$< \max|\phi'(t)| \cdot G\int_{-\tau}^{+\tau} |t|^{\alpha-1} \frac{\frac{\eta^2}{t^2} + \frac{\eta}{|t|} + 1}{\frac{\eta^2}{t^2} + \frac{1}{2}} dt \leqq \max|\phi'(t)| \frac{2GM\tau^{\alpha}}{\alpha},$$

where M is the maximum of the bounded function $(x^2 + |x| + 1)/(x^2 + \frac{1}{2})$. Thus τ can be so restricted that independently of η we have $|J_{222}(0, \eta)| < \epsilon/12$, and hence

$$|J_{222}(0, \eta') - J_{222}(0, \eta)| < \frac{\epsilon}{6}. \tag{3}$$

We now fix τ so that $|J_{21}| < \epsilon/12$, and therefore $|J_{21}(0, \eta') - J_{21}(0, \eta)| < \epsilon/6$; also so that (3), and with the proper additional restriction upon η, so that (2) shall hold; then from these three inequalities we find $|J_2(0, \eta') - J_2(0, \eta)| < \epsilon/2$ for all values of η and η' between 0 and an ascertainable positive number κ. Then as $J_1(0, \eta)$ is continuous we may, if necessary, further restrict κ so as

to obtain the inequality $|J_1(0,\eta') - J_1(0,\eta)| < \epsilon/2$, whence follows

$$\left|\frac{\partial}{\partial\xi} W(0,\eta') - \frac{\partial}{\partial\xi} W(0,\eta)\right| < \epsilon.$$

The same inequality will be found for $-\kappa < \eta, \eta' < 0$.

Hence, *$\partial W(\xi,\eta)/\partial\sigma$ approaches a finite limit as (ξ,η) approaches $[x(s), y(s)]$ along the normal to C from within; it also approaches a finite limit from without; each approach is uniform with respect to s.**

§ 4. *Generalizations.*

The desired generalizations are immediate. First, denoting by μ any fixed direction, since

$$\frac{\partial}{\partial\mu} W(\xi,\eta) = \frac{\partial W}{\partial\sigma}\cos(\sigma,\mu) + \frac{\partial W}{\partial\nu}\cos(\nu,\mu),$$

we have the result;

$\partial W(\xi,\eta)/\partial\mu$ approaches a finite limit as (ξ,η) approaches $[x(s), y(s)]$ along the normal to C from within; it also approaches a finite limit from without; both approaches are uniform with respect to s.

Secondly, the limits form continuous functions of s. Let us denote the limit of $\partial W(\xi,\eta)/\partial\mu$ by $W_m(s)$, and to fix ideas let us consider the approach from within C. Suppose $W_m(s)$ were not continuous at a point s. Then we could find a positive quantity ϵ such that

$$|W_m(s+\Delta s) - W_m(s)| > 2\epsilon, \tag{4}$$

for certain values of $|\Delta s|$ as small as we please. But the approach of $\partial W/\partial\mu$ along the normal to its limit being uniform we can so determine α that as soon as the distance of (ξ,η) from the foot of the normal is less than α, the difference between $\partial W/\partial\mu$ and its limit is less than $\epsilon/2$. That is, if

$$\begin{aligned} \xi &= x(s) - \lambda y'(s), \qquad \xi' = x(s+\Delta s) - \lambda y'(s+\Delta s), \\ \eta &= y(s) + \lambda x'(s), \qquad \eta' = y(s+\Delta s) + \lambda x'(s+\Delta s), \end{aligned} \qquad (0<\lambda<\alpha), \tag{5}$$

then

$$\left|\frac{\partial}{\partial\mu} W(\xi,\eta) - W_m(s)\right| < \frac{\epsilon}{2}, \qquad \left|\frac{\partial}{\partial\mu} W(\xi',\eta') - W_m(s+\Delta s)\right| < \frac{\epsilon}{2}, \tag{6}$$

* The limit from within will be found to be

$$\pi\phi'(s) - \int_0^l \phi'(t)\frac{\partial}{\partial n}\log\frac{1}{\rho}\,dt;$$

and from without,

$$-\pi\phi'(s) - \int_0^l \phi'(t)\frac{\partial}{\partial n}\log\frac{1}{\rho}\,dt.$$

The integrals here are absolutely convergent and are continuous in s. Cf. note, p. 46.

whence from (4), (5) and (6)

$$\left|\frac{\partial}{\partial\mu}W(\xi',\eta') - \frac{\partial}{\partial\mu}W(\xi,\eta)\right| > \epsilon \tag{7}$$

for values of $|\Delta s|$ as small as we please. But from (5) it is evident that by decreasing Δs we can bring (ξ', η') as near to (ξ, η) as we please. The inequality (7) therefore means that the derivative $\partial W/\partial\mu$ of a potential is discontinuous at a point outside the attracting masses, which is known to be untrue. Hence, *the limit $W_m(s)$ of the derivative $\partial W/\partial\mu$ in any fixed direction as (ξ, η) approaches $[x(s), y(s)]$ along the normal forms a uniformly continuous function of s; this is true for the limit from without as well as for the limit from within.*

Thirdly, we may remove the restriction that the approach be along the normal. That is, we shall show that given any positive ϵ we can find a positive r such that for all points (ξ', η') of R within the circle

$$[\xi - x(s)]^2 + [\eta - y(s)]^2 = r^2,$$

we have the inequality

$$\left|\frac{\partial}{\partial\mu}W(\xi',\eta') - W_m(s)\right| < \epsilon.$$

To do this choose α as above, then choose $\beta < \alpha/2$ so that

$$|W_m(s+\Delta s) - W_m(s)| < \frac{\epsilon}{2} \tag{8}$$

as soon as $|\Delta s| < \beta$. Then a positive quantity r can be found such that for any point (ξ', η') within the circle $[\xi - x(s)]^2 + [\eta - y(s)]^2 = r^2$ the second pair of equations (5) admit a solution $(\lambda, \Delta s)$ satisfying the inequalities $|\lambda| < \alpha$, $|\Delta s| < \beta$,* and hence the second inequality (6), and also (8), are in force. But from them we have

$$\left|\frac{\partial}{\partial\mu}W(\xi',\eta') - W_m(s)\right| < \epsilon.$$

Hence $\partial W(\xi, \eta)/\partial\mu$ *approaches its limit $W_m(s)$ for any manner of approach in R; a similar result holds for the region R' without C.*

§ 5. *The derivatives on the boundary of a potential function coinciding at interior points of R with the potential of a double distribution.*

It is well known that $W(\xi, \eta)$ is in general discontinuous at points of C, so that there can ordinarily be no question of derivatives at points of C. However, if $u(\xi, \eta)$ is a potential function coinciding with $W(\xi, \eta)$ at all interior

*This simple theorem of analysis situs admits an easy proof. The number r can be chosen independent of s.

points of R, it may be made continuous at all points of C by defining it by its limiting values, $f(s)$;* it will then have finite (in general one-sided) derivatives at each point of C, which coincide with the limits of the derivatives formed in the same direction at interior points of R. To show this, let us take the point C in question as origin, and a positive x-axis pointing into R in the given direction. Then $u(\xi, 0)$ is a continuous function of ξ in a closed ascertainable interval beginning with $(0, 0)$; it has a continuous derivative $\partial u(\xi, 0)/\partial\xi = \partial W(\xi, 0)/\partial\xi$ in the interior of the interval, and hence we may apply the law of the mean:

$$\frac{\partial}{\partial\xi}u(0, 0) = \lim_{h=0}\frac{u(h, 0) - u(0, 0)}{h} = \lim_{h=0}\frac{\partial}{\partial h}u(\vartheta h, 0) \qquad (0 < \vartheta < 1).$$

But because of the continuity of $\partial u(h, 0)/\partial h = \partial W(h, 0)/\partial h$ and because of the existence of its limit, $\partial u(0, 0)/\partial\xi$ exists and equals $W_h(s)$.

Hence *the potential function $u(\xi, \eta)$ has continuous first derivatives in the closed region R.*†

COLUMBIA, MISSOURI,
June, 1906.

* See p. 42.

† The same result holds for the potential function $u'(\xi, \eta)$ of the region R' consisting of all points outside of and upon C. To the definition of $u'(\xi, \eta)$ must then be added certain well-known restrictions regarding its behavior at infinity.

DOUBLE DISTRIBUTIONS AND THE DIRICHLET PROBLEM*

BY

O. D. KELLOGG

In 1900 FREDHOLM published his first paper on integral equations under the title *Sur une nouvelle méthode pour la résolution du problème de Dirichlet.*† He proved that if a simply connected region R of the xy-plane is given, having a boundary curve C, the coördinates of whose points considered as functions of the length of arc s of C have continuous derivatives of third order, a potential function $u(x, y)$ of R exists which has the boundary values $f(s)$ ‡ and that it can be represented as the potential of a double distribution on C. Later investigations on integral equations have been concerned either with their general theory, or with their application to differential equations other than $\Delta u = 0$.§ The object of the present paper is to establish FREDHOLM's results with considerably lighter restrictions upon the boundary curve, and also to investigate the character of the moment of the double distribution which gives the expression for the required potential function. The results will be a broader proof of the Dirichlet principle, furnishing at the same time an expression for the required potential function and in connection with my related paper *Potential Functions on the Boundary of their Region of Definition*, ‖ the establishment of simple and general conditions for the continuity at the boundary of the first derivatives of a potential function.

In the following investigation the boundary curve C, with coördinates $x(s)$, $y(s)$ expressed as functions of the arc s of C, will be subjected to the following condition:

(A) *It shall be possible to find three positive numbers* A, α *and* δ

* Presented to the Society December 1, 1906. Received for publication July 18, 1907.

† Oefversigt af Kongl. Vetenskaps-Akademiens Förhandlingar, Stockholm, 1900.

‡ FREDHOLM makes no explicit assumptions about $f(s)$. His argument implies that $f(s)$ is in general continuous and admits an absolutely convergent integral.

§ With the exception of the work of J. PLEMELJ, *Über lineare Randwertaufgaben der Potentialtheorie*, Monatshefte für Math. und Phys., vol. 15 (1904). He considers multiply connected regions, and reduces FREDHOLM's conditions upon the boundary curve to the existence of finite curvature.

‖ See this paper for notations and a discussion of the conditions used, page 40 of the present number of the Transactions.

independent of s *and of* Δs *such that* $|x'(s+\Delta s)-x'(s)|<A|\Delta s|^{\alpha}$, $|y'(s+\Delta s)-y'(s)|<A|\Delta s|^{\alpha}$, *for* $|\Delta s|<\delta$.

Let a potential function $u(x, y)$ be defined for the region R by the values $f(s)$ which it approaches on the boundary. We seek in part I to express $u(x, y)$ in the interior of R as the potential of a double distribution on C, and proceed in part II to study the character of the moment $\phi(s)$ of this double distribution as dependent upon $f(s)$.

PART I. EXPRESSION OF A POTENTIAL FUNCTION AS THE POTENTIAL OF A DOUBLE DISTRIBUTION.

§ 1. *The functional equation.*

Let us suppose that $u(x, y)$ can be expressed by means of a double distribution,

$$u(\xi, \eta)=\int_0^l \phi(t)\frac{\partial}{\partial n}\log\frac{1}{\rho}\,dt,$$

where

$$\rho^2=[\xi-x(t)]^2+[\eta-y(t)]^2.$$

The hypothesis that it approaches the boundary values $\pi f(s)$—it will be convenient to introduce the factor π—leads to the functional equation for $\phi(s)$*:

$$f(s)=\phi(s)+\int_0^l \phi(t)\frac{1}{\pi}\frac{\partial}{\partial t}\left(\arctan\frac{y(t)-y(s)}{x(t)-x(s)}\right)dt,$$

or, on introducing a parameter λ and the notation

$$K(s, t)=\frac{1}{\pi}\frac{\partial}{\partial t}\left(\arctan\frac{y(t)-y(s)}{x(t)-x(s)}\right),$$

to the equation

$$f(s)=\phi(s)+\lambda\int_0^l \phi(t)K(s, t)\,dt. \tag{1}$$

This is an "integral equation of the second kind,"† which we proceed to solve for $\phi(s)$ for the parameter value $\lambda=1$, using the methods devised by FREDHOLM.‡ Before doing so however it will be necessary to consider the function $K(s, t)$ to which the name *matrix* of the integral equation has been given,§ and to justify certain operations to be performed upon it.

*This is the usual equation of "discontinuity" for double distributions: $W_{i_0}=\pi\mu_s+W_s$. See for instance PICARD, *Traité d'Analyse*, vol. 2, p. 40.

† Cf. HILBERT: *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Göttinger Nachrichten, 1904, p. 49.

‡ Loc. cit., and *Sur une classe d'équations fonctionelles*, Acta Mathematica, vol. 27 (1903).

§ See the abstract of Professor E. H. MOORE's lectures on integral equations, *Yale Colloquium of the American Mathematical Society* (1906).

§ 2. *Operations on the matrix* $K(s, t)$.

As $K(s, t)$ has the period l in both variables we may restrict ourselves to the square $0 \leqq s \leqq l$, $0 \leqq t \leqq l$. If the indicated differentiation in $K(s, t)$ be carried out, this function will be found to be continuous in s and t except at points of the line $s = t$ and at $(0, l)$, $(l, 0)$. Application of the law of the mean and the inequalities of condition (A) show that in the neighborhood of the line $s = t$ the inequality

$$|K(s, t)| < C_0 |s - t|^{\alpha - 1} \tag{2}$$

obtains, where C_0 and α are positive constants, and a similar inequality holds at the points $(0, l)$ and $(l, 0)$.

With this inequality we may prove the possibility of changing the order of certain integrations occurring in the FREDHOLM method. The following equation is typical, all other interchanges being justified in the same manner:

$$\int_0^l \int_0^l K(s, p) K(p, q) K(q, t) dp dq = \int_0^l \int_0^l K(s, p) K(p, q) K(q, t) dq dp, \quad (s \neq t). \tag{3}$$

The corners $(0, l)$ and $(l, 0)$ we shall not consider, as they present no new difficulties. Let us consider in connection with the field of integration S a new field S' obtained by cutting out from S strips containing the lines $p = s$, $p = q$, $q = t$, and bounded by the lines $p = s \pm \epsilon$, $p = q \pm \epsilon$, and $q = t \pm \epsilon$ respectively. The equation (3) is known to hold if the iterated integral taken in either order over the absolute value of the integrand, the field of integration being S', converges to a finite limit as ϵ approaches zero. This condition is fulfilled*; hence the equation (3) holds. Similar reasoning holds for variable limits of integration, and hence may be used to justify differentiation under an integral sign provided that the integrand, after the differentiation, satisfies inequalities of the type here used, which is always the case in what follows.

Integrals of the type (3) are of frequent occurrence in the theory of integral equations, and have received the name *iterated* matrices (*iterierte Kerne*).

*To show this, divide the square s into four rectangles by the lines $p = (3s + t)/4$ and $q = (s + 3t)/4$ and justify the equation (3) for each rectangle. Take for instance that in which $0 \leqq p \leqq (3s + t)/4$, and $0 \leqq q \leqq (s + 3t)/4$. Assume $s < t$; similar reasoning holds in the reverse case. Then $|q - t| = t - q \geqq t - (s + 3t)/4 = (t - s)/4$, whence $|q - t|^{\alpha - 1} < |(t - s)/4|^{\alpha - 1}$, and therefore

$$\int\int |K(s, p) K(p, q) K(q, t)| dq dp \leqq C_0^3 |(t - s)/4|^{\alpha - 1} \int |s - p|^{\alpha - 1} \int |p - q|^{\alpha - 1} dq dp.$$

The integrals are convergent *when extended over the strips* in question and are less in value than a finite constant multiplied by ϵ^α.

Under the hypotheses upon $K(s, t)$ they are continuous * functions of s and t except at points of the line $s = t$ and the points $(0, l)$, $(l, 0)$, and this property also holds for the general iterated matrix of order n,

$$K_n(s, t) = \int_0^l K_{n-1}(s, r) K(r, t) dr, \qquad K_0(s, t) = K(s, t).$$

Moreover in the neighborhood of the line $s = t$, $K_n(s, t)$ satisfies the inequality †

$$|K_n(s, t)| < C_n |s - t|^{(n+1)\alpha - 1} \tag{4}$$

where C_n is a finite constant. It will be observed that as soon as we have $n > 1/\alpha - 1$ the functions $K_n(s, t)$ are bounded.

§ 3. *The resolvent for* $K(s, t)$.

Returning now to the integral equation (1) we observe with FREDHOLM that there exists a function $L(s, t; \lambda)$, called the "resolvent" for the matrix $K(s, t)$, such that the solution of (1) is given by

$$\phi(s) = f(s) - \lambda \int_0^l f(t) L(s, t; \lambda) dt \tag{5}$$

in all cases where λ is not a zero of a certain transcendental integral function $\delta(\lambda)$. The resolvent satisfies the equations

$$\begin{aligned} K(s, t) &= L(s, t; \lambda) + \lambda \int_0^l L(r, t; \lambda) K(s, r) dr, \\ K(s, t) &= L(s, t; \lambda) + \lambda \int_0^l L(s, r; \lambda) K(r, t) dr. \end{aligned} \tag{6}$$

Were $K(s, t)$ always finite, $L(s, t; \lambda)$ could be represented at once as the quotient of two always convergent power series in λ. In the present case, however, this is impossible and it is necessary first to find the resolvent for one of

* Given any positive number ε, we write $K_n(s, t) = J_1 + J_2$ where

$$J_1 = \int_0^{t-\tau} + \int_{s+\tau}^{t-\tau} + \int_{t+\tau}^{l} K_{n-1}(s, r) K(r, t) dr$$

and

$$J_2 = \int_{s-\tau}^{t+\tau} + \int_{t-\tau}^{s+\tau} K_{n-1}(s, r) K(r, t) dr.$$

As s is not equal to t, J_1 is continuous in s and t for fixed τ, and J_2 can be made less than $\varepsilon/4$ by restricting τ independently of Δs and Δt, provided these increments are less than an ascertainable number. Then $|\Delta J_2| < \varepsilon/2$, and because of continuity we can make $|\Delta J_1| < \varepsilon/2$.

† See FREDHOLM, Acta Mathematica, loc. cit. The inequality holds for $n < 1/\alpha - 1$. For greater values of n every $K_n(s, t)$ is finite, as may be directly verified. If $1/\alpha - 1$ is itself an integer, we may slightly decrease α, as this is evidently permissible.

the finite iterated matrices $K_n(s, t)$, and express $L(s, t; \lambda)$ in terms of it. To accomplish this, write

$$k(s, t; \lambda) = -[K(s, t) + (-\lambda) K_1(s,t) + (-\lambda)^2 K_2(s, t) + \cdots + (-\lambda)^{n-1} K_{n-1}(s, t)], \tag{7}$$

$$f(s, t; \lambda) = (-\lambda)^n K_n(s, t), \tag{8}$$

where n is the first odd integer greater than $1/\alpha - 1$, so that $f(s, t; \lambda)$ is for fixed λ a bounded function of s and t and is continuous except at points of the line $s = t$. For such points we may take it equal to zero inasmuch as in all applications it appears in integrals with respect to s or t. On the other hand $k(s, t; \lambda)$ is continuous in s and t except at points of the line $s = t$, in the neighborhood of which it satisfies an inequality $|k(s, t; \lambda)| < C(\lambda) \cdot |s-t|^{\alpha-1}$ where $C(\lambda)$ is a polynomial in λ.

The functions* $k(s, t)$ and $f(s, t)$ satisfy the equations

$$\begin{aligned} f(s, t) &= k(s, t) + K(s, t) + \lambda \int_0^l k(r, t) K(s, r)\, dr, \\ f(s, t) &= k(s, t) + K(s, t) + \lambda \int_0^l k(s, r) K(r, t)\, dr. \end{aligned} \tag{9}$$

The resolvent for $f(s, t)$ may now be formed by the FREDHOLM series. Calling it $g(s, t; \lambda)$ we have

$$g(s, t; \lambda) = \frac{\Gamma(s, t; \lambda)}{\delta(\lambda)},$$

where the transcendental integral functions Γ† and δ are given by the series

$$\begin{aligned} \Gamma(s, t; \lambda) = f(s, t; \lambda) &+ \frac{\lambda}{1!} \int_0^l \begin{vmatrix} f(r_1, r_1; \lambda), & f(r_1, t; \lambda) \\ f(s, r_1; \lambda), & f(s, t; \lambda) \end{vmatrix} dr_1 + \cdots \\ &+ \frac{\lambda^m}{m!} \int_0^l \int_0^l \cdots \int_0^l \begin{vmatrix} f(r_1, r_1; \lambda), & f(r_1, r_2; \lambda), & \cdots f(r_1, r_m; \lambda), & f(r_1, t; \lambda) \\ f(r_2, r_1; \lambda), & f(r_2, r_2; \lambda), & \cdots f(r_2, r_m; \lambda), & f(r_2, t; \lambda) \\ \cdot & \cdot & \cdot & \cdot \\ f(r_m, r_1; \lambda), & f(r_m, r_2; \lambda), & \cdots f(r_m, r_m; \lambda), & f(r_m, t; \lambda) \\ f(s, r_1; \lambda), & f(s, r_2; \lambda), & \cdots f(s, r_m; \lambda), & f(s, t; \lambda) \end{vmatrix} \\ &\times dr_1 dr_2 \cdots dr_m + \cdots, \end{aligned} \tag{10}$$

* We omit the argument λ in the notation unless it is important to call attention to the dependence on λ.

† See the FREDHOLM convergence proof, l. c., p. 367.

$$\delta(\lambda) = 1 + \frac{\lambda}{1!}\int_0^l f(r_1, r_1; \lambda)\,dr_1 + \cdots$$

$$(11) \quad + \frac{\lambda^m}{m!}\int_0^l\int_0^l \cdots \int_0^l \begin{vmatrix} f(r_1, r_1; \lambda), & f(r_1, r_2; \lambda) \cdots f(r_1, r_m; \lambda) \\ f(r_2, r_1; \lambda), & f(r_2, r_2; \lambda) \cdots f(r_2, r_m; \lambda) \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f(r_m, r_1; \lambda), & f(r_m, r_2; \lambda) \cdots f(r_m, r_m; \lambda) \end{vmatrix} dr_1 dr_2 \cdots dr_m + \cdots.$$

The quotient of (10) by (11) is also given by the series *

$$(12) \quad g(s, t; \lambda) = f(s, t; \lambda) - \lambda f_1(s, t; \lambda) + \lambda^2 f_2(s, t; \lambda) \cdots + (-\lambda)^m f_m(s, t; \lambda) + \cdots$$

where

$$f_m(s, t; \lambda) = \int_0^l f_{m-1}(s, r; \lambda) f(r, t; \lambda)\,dr, \qquad (f_0 = f).$$

Remembering the definition of $f(s, t; \lambda)$ in equation (8) we see that (10) and (11)* are uniformly convergent for any fixed λ, and that (12) is uniformly convergent for any λ less in absolute value than $1/\sqrt[n+1]{Kl}$ where K is the maximum of $|K_n(s, t)|$. It follows that these three series are continuous functions of s and t, except at points of the line $s = t$, and that they are bounded. The function $g(s, t)$ satisfies the equations

$$(13) \quad \begin{aligned} f(s, t) &= g(s, t) + \lambda \int_0^l g(r, t) f(s, r)\,dr, \\ f(s, t) &= g(s, t) + \lambda \int_0^l g(s, r) f(r, t)\,dr. \end{aligned}$$

It is now only necessary to eliminate $f(s, t)$ between (13) and (9). Suppose the value of $f(s, t)$ be substituted from (9_1) into (13_1), denoting by the subscript the first equation in each set. We obtain, if we properly change the notation for the variables of integration,

$$K(s, t) = -k(s, t) + g(s, t) + \lambda \int_0^l g(q, t) k(s, q)\,dq$$
$$+ \lambda \int_0^l \left\{ -k(r, t) + g(r, t) + \lambda \int_0^l g(q, t) k(r, q)\,dq \right\} K(s, r)\,dr.$$

Accordingly if we write

$$(14) \quad L(s, t; \lambda) = -k(s, t; \lambda) + g(s, t; \lambda) + \lambda \int_0^l g(q, t; \lambda) k(s, q; \lambda)\,dq,$$

we are led to equations (6),† which show the $L(s, t; \lambda)$ thus obtained to be a resolvent for $K(s, t)$.

* See my note in Göttinger Nachrichten for 1902, p. 165.

† The second equation (6_2) is similarly derived from (9_2) and (13_2). It is to be noted, however, that if we are to obtain the same $L(s, t; \lambda)$ in the second equation, we must have

$$\int_0^l g(s, q) k(q, t)\,dq = \int_0^l g(q, t) k(s, q)\,dq.$$

§ 4. *Solution of the functional equation.*

Using the resolvent in the form

$$L(s,t;\lambda) = -k(s,t;\lambda) + \frac{\Gamma(s,t;\lambda) + \lambda\int_0^l \Gamma(r,t;\lambda)k(s,r;\lambda)dr}{\delta(\lambda)}$$

(15)

$$= -k(s,t;\lambda) + \frac{\Gamma(s,t;\lambda) + \lambda\int_0^l \Gamma(s,r;\lambda)k(r,t;\lambda)dr}{\delta(\lambda)}$$

we attack the problem of solving the equation (1).

Let us denote by m the order of $\lambda = 1$ as a zero of $\delta(\lambda)$; in particular, m may be zero. Let m' be the index of the highest power of $\lambda - 1$ that divides

$$\Gamma(s,t;\lambda) + \lambda\int_0^l \Gamma(s,r;\lambda)k(r,t;\lambda)dr,$$

which we denote by $N(s,t;\lambda)$, so that

$$\left.\frac{N(s,t;\lambda)}{(\lambda-1)^{m'}}\right|_{\lambda=1}$$

is neither infinite nor identically zero. We assume now that $m \leqq m'$ so that $\lambda = 1$ is not a pole of $L(s,t;\lambda)$, and hence $L(s,t;1)$ is defined; it will be a continuous function of s and t except at points of the line $s = t$, where it satisfies an inequality of the same form as that to which $k(s,t;1)$ is subjected. We may now solve the equation (1) if we place upon $f(s)$ the restriction that it shall be integrable and shall admit the changes of order of integration occurring in the following.* Putting $\lambda = 1$ in equation (1), multiplying by $L(q,s;1)ds$ and integrating; then applying (6_2) for $\lambda = 1$, we have

$$\int_0^l f(s)L(q,s;1)ds = \int_0^l \phi(s)L(q,s;1)ds$$

$$+ \int_0^l\int_0^l \phi(t)K(s,t)L(q,s;1)dtds = \int^l \phi(t)K(q,t)dt,$$

and this, by (1), is $f(q) - \phi(q)$, so that,

$$\phi(s) = f(s) - \int_0^l f(t)L(s,t;1)dt. \tag{5}$$

This equation is easily verified by using the series (12) and remembering that the order of integrations involved may be changed.

*Such will be the case if $f(s)$ is continuous except at isolated points s_i in the neighborhood of which $|f(s)| < B|s - s_i|^{\beta-1}$, where B and β are positive constants.

If therefore (1) has a solution admitting the changes of order of integration involved, it must be given by (5); and conversely, the value of $\phi(s)$ in (5) satisfies (1), as may be verified by direct substitution and use of equation (6_1). Thus, on the assumption that $\lambda = 1$ is not a pole of $L(s, t; \lambda)$, we have obtained the solution of (1) and shown that the potential function $u(x, y)$ is expressible as the potential of a double distribution on C. Moreover the solution $\phi(s)$ is unique among all functions admitting the integrations and changes of order. For the difference of two solutions, $\phi_1(s) - \phi_2(s) = \psi(s)$, would satisfy the equation

$$0 = \psi(s) + \int_0^l \psi(t) K(s, t) dt,$$

and the process which derived (5) from (1), if applied to this equation, leads to the conclusion $\psi(s) = 0$. It now only remains to justify the assumption made, and show that $m \leqq m'$.

Supposing the contrary to be the case, that is, $m' < m$, the function

$$\Phi(s, t) = \frac{N(s, t; \lambda)}{\delta(\lambda)} (\lambda - 1)^{m-m'} \Big|_{\lambda=1}$$

is not identically zero; it is bounded, and also continuous except possibly at points of the line $s = t$. Multiplying (6_1) by $(\lambda - 1)^{m-m'}$ and putting $\lambda = 1$ we have

$$0 = \Phi(s, t) + \int_0^l \Phi(r, t) K(s, r) dr.$$

Integrating this with respect to t from 0 to t, and denoting

$$\int_0^t \Phi(s, t) dt$$

by $\Psi(s, t)$ we have

$$0 = \Psi(s, t) + \int_0^l \Psi(r, t) K(s, r) dr. \tag{16}$$

But this admits the interpretation: $\Psi(s, t)$ is for any fixed t the moment of a double distribution whose potential W vanishes within R. Its normal derivatives therefore also vanish, but in case the moment of a double distribution satisfies condition (B)*, its normal derivatives join on continuously to those of the potential W' of the same double distribution in the exterior region R'. Hence on applying Green's formula, W' is constant; but as it vanishes at ∞, $W' = 0$. Finally, as the moment of the double distribution multiplied by 2π is the difference of the boundary values of W and W', we have $\Psi(s, t) = 0$,

* This condition is stated in the next paragraph. That it is satisfied by $\Psi(s, t)$ will be shown later. For the theorem stated concerning the normal derivatives of a double distribution see my paper *Potential functions*, previously cited, p. 44.

whence $\Phi(s, t) = 0$. This proves the falsity of the assumption $m' < m$. We may therefore state the following theorem:

Given a function $f(s)$ of the arc of C, continuous except at isolated points s_i in the neighborhood of which $|f(s)| < B|s - s_i|^{\beta-1}$, where B and β are positive constants; there exists a potential function $u(x, y)$ of R which approaches $f(s)$ at all points where $f(s)$ is continuous, and $u(x, y)$ may be represented in the interior of R as the potential of a double distribution on C; moreover the double distribution is uniquely determined by the demand that its moment satisfy the same continuity requirements as $f(s)$.*

Part II. The character of the moment $\phi(s)$.

In my paper cited above it was shown that a potential function $u(x, y)$ of R which coincides within R with the potential of the double distribution on C,

$$W(\xi, \eta) = \int_0^l \phi(t) \frac{\partial}{\partial n} \log \frac{1}{\rho} dt,$$

and which is defined on the boundary by its limiting values, has continuous first derivatives in the closed region R provided the boundary C satisfies conditions (A), and $\phi(s)$ satisfies the following condition:

(B) *The function $\phi'(s)$ is continuous, and the integral*

$$\int_0^\tau \left| \frac{\phi'(s+t) - \phi'(s-t)}{t} \right| dt,$$

approaches 0 uniformly with respect to s as τ approaches 0.

It was further stated, the proof being deferred to the present paper, that if $u(x, y)$ is defined by boundary values $f(s)$ satisfying condition (B), it can be represented as the potential of a double distribution on C; and moreover that the moment $\phi(s)$ of the double distribution satisfies the same condition (B). We have now justified the first part of this statement, and it remains to prove the part concerning the moment. At the same time we shall see that the function $\Psi(s, t)$ for fixed t has the same character as $\phi(s)$, a fact essential to the reasoning in the preceding page.

A study of equation (5) and of the function $L(s, t; \lambda)$ reveals the continuity of $\phi'(s)$ and $\Psi'_s(s, t)$ (with possible exception of isolated points, near which the functions remain finite); by means of this fact and equations (1) and (6), we shall show that $\phi(s)$ and $\Psi(s, t)$ satisfy (B).

* The generality of this as an existence proof depends upon the lightness of condition (A).

§ 1. *Continuity of the functions involved.*

It is not evident that the equation (5) can be differentiated with respect to s, a process necessary to the study of $\phi'(s)$. Its form must first be changed by integration by parts. In doing this we meet with the function

$$\frac{\partial}{\partial s}\int_0^t L(s, t; \lambda)dt,$$

which must be investigated first.

Let us suppose that we are measuring s from a point where the tangent to C is parallel with the x-axis. Then the function

$$\Theta(s, t) = \frac{1}{\pi}\arctan\frac{y(s) - y(t)}{x(s) - x(t)}$$

may be made one-valued in the region $0 \leqq s < l$, $0 \leqq t < l$ by the demand that it pass continuously over into 0 as s and t approach 0.* We have then the following relations:†

$$\Theta(s, t) = \Theta(t, s), \qquad \Theta(s, l) - \Theta(s, 0) = \Theta(l, t) - \Theta(0, t) = 1$$

$$\frac{\partial}{\partial t}\Theta(s, t) = K(s, t), \qquad \frac{\partial}{\partial s}\Theta(s, t) = K(t, s)$$

(17)
$$\int_0^t K(s, t)dt = \Theta(s, t) - \Theta(s, 0), \qquad \int_0^l K(s, t) = 1$$

$$\frac{\partial}{\partial s}\int_0^t K(s, t)\,dt = K(t, s) - K(0, s).$$

We seek an equation similar to the last for

$$\frac{\partial}{\partial s}\int_0^t L(s, t; \lambda)dt.$$

Defining the continuous ‡ function

$$\Theta_1(s, t) = \int_0^t K(s, r)\Theta(r, t)dt,$$

we find that

$$\Theta_1(s, l) - \Theta_1(s, 0) = 1, \qquad \Theta_1(l, t) - \Theta_1(0, t) = 0,$$

and

$$\frac{\partial}{\partial t}\Theta_1(s, t) = K_1(s, t).$$

* AMES, *Note on the orientation of a secant*, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 240.

† It is to be understood that $\Theta(s, l) = \lim_{t=l}\Theta(s, t)$, $\qquad \Theta(l, t) = \lim_{s=l}\Theta(s, t)$.

‡ That $\Theta_1(s, t)$ is continuous may be proven as in the footnote, p. 54.

To obtain $\partial\Theta_1(s,t)/\partial s$ integrate first by parts in the equation defining $\Theta_1(s,t)$. The result is

$$\Theta_1(s,t) = 1 + \Theta(s,0) + \Theta(0,t) - \int_0^t \Theta(s,r)K(t,r)dr,$$

whence

$$\frac{\partial}{\partial s}\Theta_1(s,t) = K(0,s) - K_1(t,s),$$

$$\int_0^t K_1(s,t)dt = \Theta_1(s,t) - \Theta_1(s,0),$$

and hence

$$\frac{\partial}{\partial s}\int_0^t K_1(s,t)\,dt = -\,[K_1(t,s) - K_1(0,s)].$$

Proceeding in a similar manner we obtain by mathematical induction the formula for the general iterated matrix:

$$\frac{\partial}{\partial s}\int_0^t K_i(s,t)dt = (-1)^i[K_i(t,s) - K_i(0,s)]. \tag{18}$$

With these results equations (7), (8) and (12) give

$$\frac{\partial}{\partial s}\int_0^t k(s,t;\lambda)dt = k(t,s;-\lambda) - k(0,s;-\lambda),$$

$$\frac{\partial}{\partial s}\int_0^t f(s,t;\lambda)dt = f(t,s;-\lambda) - f(0,s;-\lambda),$$

$$\frac{\partial}{\partial s}\int_0^t g(s,t;\lambda)dt = g(t,s;-\lambda) - g(0,s;-\lambda),$$

and as $\delta(\lambda) = \delta(-\lambda)$,*

$$\frac{\partial}{\partial s}\int_0^t \Gamma(s,t;\lambda)dt = \Gamma(t,s;-\lambda) - \Gamma(0,s;-\lambda).$$

The analogous equation holds for the function

$$\lambda\int_0^t \Gamma(r,t;\lambda)k(s,r;\lambda)dr \equiv \lambda\delta(\lambda)\int_0^t g(r,t;\lambda)k(s,r;\lambda)dr,$$

whence by (15) the desired result,

$$\frac{\partial}{\partial s}\int_0^t L(s,t;\lambda)dt = L(t,s;-\lambda) - L(0,s;-\lambda) \tag{19}$$

is obtained.

* Equations (8) and (11) show $\delta(\lambda)$ to be a power series in λ^{n+1}; the grounds for taking n odd (p. 55) are thus apparent.

Finally we must consider for $m' < m$ the expression

$$[L(s, t, \lambda)(\lambda-1)^{m-m'}]_{\lambda=1} = \Phi(s, t).$$

The development of the integral transcendental functions involved about the point $\lambda = 1$ leads to the result

$$\frac{\partial}{\partial s}\int_0^t [L(s, t; \lambda)(\lambda-1)^{m-m'}]_{\lambda=1} dt = [\{L(t, s; -\lambda) - L(0, s; -\lambda)\}(\lambda-1)^{m-m'}]_{\lambda=1}. \tag{20}$$

This function, which is none other than $\Psi'_s(s, t)$, appears as a uniformly convergent series of iterated matrices, all of order n or higher. It is therefore continuous except possibly at points of the lines $s = t$ and $s = 0$, in the neighborhood of which it is finite, so that we may state the result:

$\Psi'_s(s, t)$ is bounded, and for fixed t it is a continuous function of s except possibly at the points $s = t$ and $s = 0$.

In the case $m \leqq m'$, for which $\lambda = 1$ is not a pole of $L(s, t; \lambda)$, we may use equation (17) for $\lambda = 1$. Then integrating by parts in equation (5) we have

$$\phi(s) = f(s) - \left[f(t)\int_0^t L(s, t; 1)dt\right]_{t=0}^{t=l} + \int_0^l f'(t)\int_0^t L(s, t; 1)dt$$

$$= f(s) - f(l)\int_0^l L(s, t; 1)dt + \int_0^l f'(t)\int_0^t L(s, t; 1)dt.$$

If now the equation (6_2) for $\lambda = 1$ be integrated with respect to t from 0 to l, we have by (17),

$$\int_0^l L(s, t; 1)dt = \tfrac{1}{2}.*$$

Hence differentiating the above equation for $\phi(s)$, we obtain

$$\phi'(s) = f'(s) + \int_0^l \left[f'(t)\frac{\partial}{\partial s}\int_0^t L(s, t; 1)dt\right]dt$$

$$= f'(s) + \int_0^l f'(t)L(t, s; -1)dt - L(0, s; -1)\int_0^l f'(t)dt$$

$$= f'(s) + \int_0^l f'(t)L(t, s; -1)dt,$$

which proves that $\phi'(s)$ *is continuous.*†

* It is interesting to note that this equation precludes the possibility $m < m'$. For then $L(s, t; 1) = -k(s, t; 1)$, and from the definition of $k(s, t; \lambda)$, we see that

$$\int_0^l k(s, t; 1)dt$$

is an integer.

† The development of the integral transcendental functions involved about $\lambda = 1$, shows that $L(t, s; -1)$ is equal to $-k(t, s; -1)$ plus a uniformly convergent series of iterated matrices

§ 2. *The functions* $\phi(s)$ *and* $\Psi(s, t)$ *satisfy conditions* (B).

Integrating by parts in equations (1) and (16) and then differentiating with respect to s, we have

$$f'(s) = \phi'(s) - \int_0^l \phi'(t) K(t, s) dt,$$

$$0 = \psi_s'(s, t) - \int_0^l \psi_r'(r, t) K(r, s) dr.$$

If now two functions satisfy condition (B), their difference does. Hence as $f(s)$ does, it is only necessary to show that the two integrals of the last equations do. This is included in the following theorem which we proceed to prove.

If $g(s)$ *is bounded, and is continuous except at isolated points, the function*

$$I(s) = \int_0^l g(t) K(t, s) dt,$$

where $K(t, s)$ *is the function defined above, is uniformly algebraically continuous, that is, satisfies the hypothesis upon* $x'(s)$ *and* $y'(s)$ *in condition* (A).

For convenience let us take as origin for the variable s the point to be considered. We are to show that we can determine three positive constants, C', α', and δ', independent of s and Δs, such that

$$|I(s + \Delta s) - I(s)| < C' |\Delta s|^{\alpha'}$$

as soon as $|\Delta s| < \delta'$; or in particular, for the point in question,

$$|I(s) - I(0)| < C' |s|^{\alpha'}$$

as soon as $|s| < \delta'$. To do this, write $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{-\sqrt{s}}^{\sqrt{s}} g(t) K(t, s) dt, \qquad I_2 = \int_{-\eta}^{-\sqrt{s}} + \int_{\sqrt{s}}^{\eta} g(t) K(t, s) dt,$$

and

$$I_3 = \int_{-\frac{l}{2}}^{-\eta} + \int_{\eta}^{\frac{l}{2}} g(t) K(t, s) dt,$$

leaving the number η to be specified presently. We have supposed $s \geqq 0$; the reasoning for $s \leqq 0$ is exactly similar.

Considering $I_1(s)$ with the inequality (2) for $K(t, s)$, we have

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of order n or higher; that is, plus a function of s and t which is continuous except possibly at points of the line $s = t$, in the neighborhood of which it is bounded. Reasoning like that of the footnote, p. 54, then demonstrates the asserted continuity of $\phi'(s)$.

$$|I_1(s)| < \max|g(t)| C_0 \int_{-\sqrt{s}}^{\sqrt{s}} |s-t|^{\alpha-1} dt < C_1' s^{\frac{\alpha}{2}}$$

where C_1' is a finite constant. Then as $I_1(0) = 0$, we have

$$|I_1(s) - I_1(0)| < C_1' s^{\frac{\alpha}{2}}. \tag{21}$$

For $I_2(s)$ we have

$$I_2(s) - I_2(0) = \int_{-\eta}^{-\sqrt{s}} + \int_{\sqrt{s}}^{\eta} g(t) K(t, s)\, dt - \int_{-\eta}^{\eta} g(t) K(t, 0)\, dt$$

$$= \int_{-\eta}^{-\sqrt{s}} + \int_{\sqrt{s}}^{\eta} g(t)[K(t, s) - K(t, 0)]\, dt - \int_{-\sqrt{s}}^{\sqrt{s}} g(t) K(t, 0)\, dt,$$

where the last integral satisfies the above inequality for $|I_1(s)|$. Hence

$$|I_2(s) - I_2(0)| < \max|g(t)| \left\{ \int_{-\eta}^{-\sqrt{s}} + \int_{\sqrt{s}}^{\eta} |K(t, s) - K(t, 0)|\, dt \right\} + C_1' s^{\frac{\alpha}{2}}. \tag{22}$$

The sum of the two integrals, in which $\sqrt{s} \leqq |t| \leqq \eta$, will be found less than a finite constant times $s^{\alpha/2}$. To show this we study the common integrand, writing

$$K(t, s) - K(t, 0) = N(t, s) / \pi \rho^2 \rho_0^2,$$

$$\rho^2 = [x(t) - x(s)]^2 + [y(t) - y(s)]^2, \qquad \rho_0^2 = [x(t) - x(0)]^2 + [y(t) - y(0)]^2,$$

$$N(t, s) = \rho_0^2 \{ [x(s) - x(t)]\, y'(s) - [y(s) - y(t)]\, x'(s) \}$$

$$- \rho^2 \{ [x(0) - x(t)]\, y'(0) - [y(0) - y(t)]\, x'(0) \}.$$

The law of the mean applied to the differences involved gives $\rho^2 = (s-t)^2 F(s, t)$, $\rho_0^2 = t^2 F(0, t)$, where $F(s, t)$ has a positive lower limit $\underline{F}$, so that $\rho^2 \geqq (s-t)^2 \underline{F}$ and $\rho_0^2 \geqq t^2 \underline{F}$. To obtain therefore an effective upper limit for the integral of (22) we must arrange $N(t, s)$ so as to expose the factors $(s - t)$, s, t or fractional powers of them. We may do this by writing $N = N_1 + N_2 + N_3$, where

$$N_1 = \rho_0^2 \{ [y'(s) - y'(0)]\, [x(s) - x(t)] - [x'(s) - x'(0)]\, [y(s) - y(t)] \},$$

$$N_2 = \rho_0^2 \{ [x(s) - x(0)]\, y'(0) - [y(s) - y(0)]\, x'(0) \},$$

$$N_3 = (\rho_0^2 - \rho^2) \{ [x(0) - x(t)]\, y'(0) - [y(0) - y(t)]\, x'(0) \}.$$

The law of the mean together with condition (A) leads, provided 2η is less than δ of that condition, to the equations

$$x'(s)-x'(0)=F\cdot s^{\alpha}, \qquad y'(s)-y'(0)=F\cdot s^{\alpha},$$
$$x(s)-x(t)=F\cdot(s-t), \qquad y(s)-y(t)=F\cdot(s-t),$$
$$x(s)-x(0)=s[x'(0)+F\cdot s^{\alpha}], \qquad y(s)-y(0)=s[y'(0)+F\cdot s^{\alpha}],$$

where F denotes various bounded functions. These give

$$N_1=\rho_0^2 F\cdot s^{\alpha}|s-t|, \qquad N_2=\rho_0^2 F\cdot s^{\alpha+1},$$

$$\rho_0^2-\rho^2=-[x(s)-x(0)][x(s)+x(0)-2x(t)]$$
$$-[y(s-y(0)][y(s)+y(0)-2y(t)]=F\cdot s\{|s-t|+|t|\},$$

whence

$$N_3=F\cdot s|t|^{\alpha+1}\{|s-t|+|t|\}.$$

Using these values and denoting by F a constant greater than any $F/\underline{F}$, we have

$$|K(t,s)-K(t,0)|<\overline{F}s^{\frac{\alpha}{2}}\left\{s^{\frac{\alpha}{2}}\left(\frac{1}{|s-t|}+\frac{s}{(s-t)^2}\right)+s^{1-\frac{\alpha}{2}}|t|^{\alpha-1}\left(\frac{1}{|s-t|}+\frac{|t|}{(s-t)^2}\right)\right\}.$$

The quantity in braces will be found less than a linear function of $|t|^{\alpha-1}$. For $|t|\geqq\sqrt{s}$, whence $|t|\geqq t^2\geqq s$ if $|t|<1$, which we can effect by choosing $\eta\leqq 1$; hence $|s-t|\geqq|t|-s\geqq|t|-t^2=|t|(1-|t|)$. Further, on restricting η to be less than $\frac{1}{2}$, this gives $|s-t|>|t|/2$. Whence

$$\frac{s^{\frac{\alpha}{2}}}{|s-t|}<\frac{2s^{\frac{\alpha}{2}}}{|t|}\leqq\frac{2|t|^{\alpha}}{|t|}=2|t|^{\alpha-1}.$$

Similarly

$$\frac{s^{1+\frac{\alpha}{2}}}{|s-t|^2}<4t^{\alpha}, \qquad \frac{s^{1-\frac{\alpha}{2}}|t|^{\alpha-1}}{|s-t|}<2, \qquad \frac{s^{1-\frac{\alpha}{2}}|t|^{\alpha}}{|s-t|^2}<4.$$

Hence,

$$|K(t,s)-K(t,0)|<\overline{F}s^{\frac{\alpha}{2}}\{10+2|t|^{\alpha-1}\}.$$

With this the inequality (22) becomes

$$|I_2(s)-I_2(0)|<C_2's^{\frac{\alpha}{2}}. \tag{23}$$

Finally, consider $I_3(s)$. We restrict δ to be less than $\eta/2$ and hence it follows that $|s-t|>\eta/2$, $|t|$ being greater than η, so that the rational function $K(t,s)$ of uniformly algebraically continuous functions of s has a denominator with positive lower limit, and is hence itself uniformly algebraically continuous with the same index α. Consequently for suitably chosen C_3',

$$|I_3(s)-I_3(0)|<C_3's^{\alpha}\leqq C_3's^{\frac{\alpha}{2}}. \tag{24}$$

From the inequalities (21), (23) and (24) there results the inequality

$$|I(s)-I(0)|<C's^{\alpha'},$$

in which $C'=C_1'+C_2'+C_3'$ and $\alpha'=\alpha/2$, where C' as well as α' and δ' are independent of the particular point s investigated. Accordingly we may write the inequality

$$|I(s+\Delta s)-I(s)|<C'|\Delta s|^{\alpha'}$$

as obtaining for every s, as soon as $|\Delta s|<\delta'$, and the above stated theorem concerning $I(s)$ is established.

Thus $\Psi(s,t)$ for fixed t satisfies condition (B), and the assumption of page 58 is justified, and consequently the case for which $\lambda=1$ is a pole of $L(s,t;\lambda)$ is impossible. For $\phi(s)$ we may state the theorem:

If the boundary values $f(s)$ of a potential function $u(x,y)$ satisfy condition (B), so also does the moment $\phi(s)$ of the double distribution on C whose potential coincides with $u(x,y)$ within R.

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GROUPS DEFINED BY THE ORDERS OF TWO GENERATORS AND THE ORDER OF THEIR COMMUTATOR*

BY

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§ 1. *Introduction.*

The commutator of two operators (s_1, s_2) may be represented by $s_1^{-1}s_2^{-1}s_1s_2$. If the four elements of this commutator are permuted in every possible manner there result 24 operators. Sixteen of these are equal to the identity irrespective of the choice of s_1, s_2, while the other eight are commutators and may all be distinct.† These eight have the following forms:

$$s_1^{-1}s_2^{-1}s_1s_2,\ s_2^{-1}s_1s_2s_1^{-1},\ s_2s_1^{-1}s_2^{-1}s_1,\ s_1s_2s_1^{-1}s_2^{-1};$$

$$s_2^{-1}s_1^{-1}s_2s_1,\ s_1s_2^{-1}s_1^{-1}s_2,\ s_1^{-1}s_2s_1s_2^{-1},\ s_2s_1s_2^{-1}s_1^{-1}.$$

The last four are the inverses of the first four, and these are the transforms of the first with respect to $1, s_1^{-1}, s_2^{-1}, s_2^{-1}s_1^{-1}$, respectively. In particular, *the eight commutators which may be obtained from a single commutator by permuting its elements are of the same order, and if this order is two not more than four of them are distinct.* While the order of a commutator is independent of the particular form by which it is represented, we shall always use the first one of these forms to represent the commutator of s_1 and s_2. Its inverse represents the commutator of s_2 and s_1.

It is sometimes convenient to employ the order of the commutator in the abstract definition of a group. The simplest instance with respect to non-abelian groups is exhibited by the following theorem: *If any two operators of order two have a commutator of even order they generate a group whose order is four times the order of their commutator.* This theorem follows directly from the facts that two operators of order 2 generate a dihedral group and that the commutator of two operators of order 2 is the square of their product. From these facts it follows also that two operators of order 2 whose commutator is of odd order generate a group whose order is either twice or four times the order of their commutator.

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† Bulletin of the American Mathematical Society, vol. 5 (1899), p. 239.

The next case, in order of simplicity, is when one of the generating operators (s_1) is of order 2, the other (s_2) of order 3 and their commutator of order 2. These conditions may be expressed as follows: $s_1^2 = s_2^3 = (s_1 s_2^2 s_1 s_2)^2 = 1$. Since s_1 and $s_2^2 s_1 s_2$ are of order 2 and their product is also of this order, they are commutative and 1, s_1, $s_2^2 s_1 s_2$, $s_1 s_2^2 s_1 s_2$ is the four-group. The other transform of s_1 with respect to the powers of s_2, viz., $s_2 s_1 s_2^2$, is commutative with each operator of this four-group since

$$s_1 s_2 s_1 s_2^2 = s_2 s_1 s_2^2 s_1 \text{*} \text{ and } s_2 s_1 s_2^2 \cdot s_2^2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 = s_2^2 s_1 s_2^2 s_1 s_2^2 = s_2^2 s_1 s_2 \cdot s_2 s_1 s_2^2.$$

From this it follows that s_1, $s_2^2 s_1 s_2$, $s_2 s_1 s_2^2$ generate either the abelian group of order 8 and of type $(1, 1, 1)$ or a subgroup of order 4. As this group contains s_1 and is transformed into itself by s_2 it is invariant under the group $\{s_1, s_2\}$ generated by s_1 and s_2. If the order of this group is 8 one of the seven subgroups of order 4 which are contained in it is transformed into itself by s_2, and s_2 could not be commutative with the four operators of this subgroup,† hence it follows that $\{s_1, s_2\}$ contains the tetrahedral group. If it contains any additional operators it must include invariant operators, and, as its order cannot exceed 24, it must be the direct product of the tetrahedral group and the group of order 2. Hence the theorem: *If an operator of order* 2 *and an operator of order* 3 *have a commutator of order* 2 *they generate either the tetrahedral group or the direct product of this group and the group of order* 2.

It has now been proved that if s_1, s_2 satisfy the following conditions:

$$s_1^2 = 1, \qquad s_2^n = 1, \qquad (s_1 s_2^{-1} s_1 s_2)^2 = 1,$$

the group $\{s_1, s_2\}$ is entirely determined by the value of n when $n = 2$; and when $n = 3$, $\{s_1, s_2\}$ may be one of two groups. It will now be proved that $\{s_1, s_2\}$ may be one of an infinite system of groups whenever $n > 3$. For instance, when $n = 4$, we may let

$$s_1 = a_3 b_1 \cdot b_2 c_1 \cdot c_2 d_1 \cdot \cdots,$$
$$s_2 = a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 d_3 d_4 \cdot \cdots,$$
$$s_2^{-1} s_1 s_2 = a_4 b_2 \cdot b_4 c_2 \cdot c_4 d_2 \cdot \cdots.$$

Since s_1, $s_2^{-1} s_1 s_2$ are commutative the commutator $s_1 s_2^{-1} s_1 s_2$ is of order 2. The group generated by s_1, s_2 is transitive and hence its order is a multiple of its degree. As this degree can be increased without limit, s_1, s_2 may be so selected that the order of $\{s_1, s_2\}$ exceeds any given number. It is evident that s_1 and s_2 can be chosen in a similar manner whenever $n > 3$, so that we have the

* These operators are of order 2 since they are the transforms of operators of order two.

† MOORE, Bulletin of the American Mathematical Society, vol. 1 (1894), p. 61.

result: *If an operator of order* 2 *and an operator of order* n *have a commutator of order* 2, *if* $n > 3$, *these operators may be so selected that they generate any one of an infinite system of groups of finite order.*

§ 2. *The groups generated by two operators of order three whose commutator is of order two.*

We shall suppose that s_1 and s_2 satisfy the following conditions:

$$s_1^3 = s_2^3 = 1, \qquad s_1^2 s_2^2 s_1 s_2 = s_2^2 s_1^2 s_2 s_1,$$

and shall first consider the group (H) generated by the four commutators

$$s_1^2 s_2^2 s_1 s_2, \qquad s_2^2 s_1 s_2 s_1^2, \qquad s_2 s_1^2 s_2^2 s_1, \qquad s_1 s_2 s_1^2 s_2^2.$$

From the fact that two operators of order 3 whose product is of order 2 generate the tetrahedral group, it follows that each of these operators is commutative with two of the other three. For instance, $s_1^2 s_2^2 s_1 s_2$ is commutative with $s_2^2 s_1 s_2 s_1^2$ and also with $s_2 s_1^2 s_2^2 s_1$ since each of the two tetrahedral groups generated by $s_2^2 s_1 s_2$, s_1^2 and s_2, $s_1^2 s_2^2 s_1$, respectively, contains $s_1^2 s_2^2 s_1 s_2$ together with one or the other of the two commutators in question. Similarly, it may be observed that $s_2^2 s_1 s_2 s_1^2$ is commutative with $s_1^2 s_2^2 s_1 s_2$ and $s_1 s_2 s_1^2 s_2^2$; $s_2 s_1^2 s_2^2 s_1$ with $s_1 s_2 s_1^2 s_2^2$ and $s_1^2 s_2^2 s_1 s_2$; $s_1 s_2 s_1^2 s_2^2$ with $s_2^2 s_1 s_2 s_1^2$ and $s_2 s_1^2 s_2^2 s_1$. Hence the two dihedral groups $D_1 = \{ s_1^2 s_2^2 s_1 s_2,\ s_1 s_2 s_1^2 s_2^2 \}$ and $D_2 = \{ s_2^2 s_1 s_2 s_1^2,\ s_2 s_1^2 s_2^2 s_1 \}$ have each operator of the one commutative with every operator of the other. These two groups generate H.

The group H is an invariant subgroup of $G = \{ s_1, s_2 \}$ since H is transformed into itself by s_1 and s_2 by virtue of the following equations:

$$s_1 s_2^2 s_1 s_2 s_1 = (s_1^2 s_2^2 s_1 s_2 \cdot s_2^2 s_1 s_2 s_1^2)^{-1}, \qquad s_1^2 s_2 s_1^2 s_2^2 s_1^2 = s_1^2 s_2 s_1 s_2^2 \cdot s_2 s_1 s_2^2 s_1^2,$$

$$s_2^2 s_1^2 s_2^2 s_1 s_2^2 = s_2^2 s_1^2 s_2 s_1 \cdot s_1^2 s_2 s_1 s_2^2, \qquad s_2 s_1 s_2 s_1^2 s_2 = s_2 s_1 s_2^2 s_1^2 \cdot s_1 s_2^2 s_1^2 s_2.$$

As H is solvable, the commutator subgroup of G is solvable and hence G is solvable. If one of the two dihedral groups D_1, D_2 is abelian the other is also abelian, since the two generators of such an abelian group would be commutative with every operator of H and hence this would also be true of their transforms. As a generator of the other dihedral group would be such a transform, the statement is proven.

If H contains any operators of odd order the totality of these together with the identity must constitute an abelian characteristic subgroup of H. Hence neither s_1 nor s_2 can occur in H; and s_1, H generate a group whose order is three times that of H. This group is transformed into itself by s_2 since it involves $s_2^2 s_1 s_2$. The order of G is therefore the order of H multiplied by either

3 or 9, and the commutator quotient group* has one of these two numbers for its order. The smallest order that G can have is 12. If it has this order it is the tetrahedral group and we may let $s_1 = abc$, $s_2 = acd$ and then $s_1^2 s_2^2 s_1 s_2 = ab \cdot cd$. The order of H could not be less than 4, since a group whose order is twice an odd number contains a subgroup composed of all its operators of odd order together with the identity.

When H is abelian its order cannot exceed 16 and hence the order of G cannot exceed 144 in this case. A group of order 144 in which H is abelian may be generated as follows: Let

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \quad \text{and} \quad s_2 = adg \cdot ejh \cdot fki.$$

The four given commutators will then be in order:

$$ad \cdot be \cdot gl \cdot hj, \qquad ad \cdot cf \cdot gl \cdot ik, \qquad ag \cdot bh \cdot dl \cdot ej, \qquad ag \cdot ci \cdot dl \cdot fk.$$

These four substitutions illustrate that the four commutators given in our first theorem may be distinct and they may be used as the four independent generators of the abelian group of order 16 and of type (1, 1, 1, 1). As s_1 and H generate a transitive group of order 48 while H involves three systems of intransitivity, and hence the average number of letters in all its substitutions is 9, all the substitutions of this group of order 48 which are not in H must be of degree 12. Otherwise the average number of letters in all its substitutions would not be 11. Since s_2 is not in this group of order 48 the order of G is actually 144. That is, there is a G in which H has its maximal order as an abelian group and the order of G is 144.

When H is an abelian group of order 16 the order of G cannot be 48, since any two non-commutative substitutions (t_1, t_2) of order 3 in such a group would have the property that either $t_1 t_2$ or $t_1 t_2^2$ would be of order 2, and hence they would generate the tetrahedral group. Moreover H cannot be abelian and of order 8, since either s_1 or s_2 and this group of order 8 and of type (1, 1, 1) would generate the direct product of the tetrahedral group and the group of order 2. As this group of order 24 contains a characteristic tetrahedral subgroup involving all its operators of order 3, it is not generated by two operators of order 3, nor is a group of order 72 which contains it invariantly generated by one of its operators of order 3 and an operator of order 3 which is not found in the invariant subgroup. This proves the theorem: *If H is abelian the order of G is* 12, 36, *or* 144. We gave an example where the order of G is 12 and also one where this order is 144. It is easy to see that the following substitutions generate such a group of order 36:

* The commutator quotient group is the quotient group with respect to the commutator subgroup.

$s_1 = abc \cdot def \cdot ghi \cdot jkl$, $s_2 = adg \cdot ble \cdot cij$, $s_1^2 s_2^2 s_1 s_2 = ak \cdot bh \cdot cf \cdot dg \cdot el \cdot ij$,

$s_2 s_1^2 s_2^2 s_1 = ad \cdot ci \cdot eh \cdot bl \cdot gk \cdot fj$, $s_2^2 s_1 s_2 s_1^2 = ag \cdot be \cdot cj \cdot dk \cdot fi \cdot hl$, $s_1 s_2 s_1^2 s_2^2 = s_1^2 s_2^2 s_1 s_2$.

This group of order 36 contains $abc \cdot dli \cdot ejg \cdot fkh$ invariantly and permutes its cycles according to the alternating group of degree 4. Hence G is the direct product of the tetrahedral group and the group of order 3. It is not difficult to see that G is always such a direct product when its order is 36, since it must contain a tetrahedral group invariantly. That is, when H is of order 4, G is either the tetrahedral group or it is the direct product of this group and the group of order 3. In other words, there is one and only one group of each of the orders 12 and 36 which is generated by two operators of order 3 whose commutator is of order 2.

When H is non-abelian both of the dihedral subgroups D_1, D_2 are non-abelian. We shall now prove that the order of D_1 cannot be twice an odd number. If this were the case, D_1 would involve no invariant operator besides the identity and hence H would be the direct product of D_1 and D_2. The odd operators of H (together with the invariant operator of order 2 in D_2 if such an operator is present) would generate an invariant subgroup of G. The corresponding quotient group would be abelian since its operators of order 2 are invariant under it. This is impossible since H was supposed to be generated by the commutators of G. Hence *the order of each of the subgroups* D_1, D_2 *is divisible by* 4 *whenever* H *is non-abelian.*

It will now be proved that when H is non-abelian D_1 and D_2 have the same order. This follows directly from the facts that the order of each of these dihedral groups is divisible by 4, that one of the two generators of order 2 of one of them can be transformed into one of the two generators of the other, and that each of the operators of D_1 is commutative with every operator of D_2. It is now easy to see that H is not the direct product of D_1 and D_2. Such a direct product would contain the direct product of the maximal cyclic subgroups in D_1 and D_2 as a characteristic subgroup, and the quotient group of G with respect to this characteristic subgroup would be abelian. The latter direct product is a characteristic subgroup of G, since it is generated by all the operators of highest order in H which are commutative with half the operators of H.

The case which remains to be considered is the one in which D_1 and D_2 are non-abelian and have two common operators when the order of D_1 exceeds 8. The operators of highest order in these subgroups would again generate a characteristic subgroup of G since they include all of the operators of H which have this order and are commutative with half of its operators. As the quotient group with respect to this characteristic subgroup would be abelian, we have the important result: If two operators of order 3 have a commutator of order 2

they generate a group whose order is one of the five numbers, 12, 36, 144, 96, 288. This order, however, could not be 96. If G were of order 96 it would be isomorphic with the given group of order 144. Since not more than three operators of this G_{144} could correspond to the identity in G_{96} and as the isomorphism could not be $(\alpha, 1)$, it follows that G_{144} would contain an invariant subgroup of order 3. As the corresponding quotient group could not be generated by two operators of order 3, whose commutator is of order 2, this is impossible. That is, it is not possible to generate a group of order 96 by two operators of order 3 whose commutator is of order 2.

In the preceding paragraph we proved that the substitution group of order 144 which was constructed above does not contain an invariant subgroup of order 3. It is possible to prove a much more general theorem, viz.: *If a group of order* 144 *is generated by two operators of order* 3 *whose commutator is of order* 2, *it is the direct product of two tetrahedral groups.* Since such a group contains the abelian subgroup of order 16 and of type (1, 1, 1, 1) invariantly and this subgroup involves 35 subgroups of order 4, the group contains at least two invariant subgroups of order 4. If these had a common operator of order 2, G would contain an invariant subgroup of order 8. This is impossible since the corresponding quotient group could not be generated by two operators of order 3. Hence every such group of order 144 contains two invariant subgroups of order 4 and these two subgroups have only the identity in common. With respect to these subgroups the quotient group of G is the direct product of the tetrahedral group and the group of order 3, since this quotient is generated by two operators of order 3 whose commutator is of order 2. Hence G contains two invariant subgroups of order 12 which have at most three common operators. If they had three such operators these would constitute an invariant subgroup of G. This is impossible since the corresponding quotient group could not be generated by two operators of order 3 having a commutator of order 2. Hence G is the direct product of two alternating groups of order 12.

It has now been proved that there is one and only one group of each of the orders 12, 36 and 144 which is generated by two operators of order 3 having a commutator of order 2. It has also been proved that if another such group exists its order is 288 and it must have a $(2, 1)$ isomorphism with the direct product of two alternating groups. Moreover, its H is generated by two octic groups having each operator of the one commutative with every operator of the other and having two operators in common. As a transitive group, H may therefore be generated by the regular octic and its associate,* or by the regular quaternion group and its associate, since this group may be generated in either of these two ways. This suggests that G may be generated by the regular

*Quarterly Journal of Mathematics, vol. 28 (1896), p. 249.

group of order 24 which does not involve a subgroup of order 12 and its associate. From this fact we readily obtain the following relations:

$$s_1 = ark \cdot bxn \cdot cwl \cdot dqm \cdot evo \cdot ftj \cdot gsp \cdot hui,$$

$$s_2 = adc \cdot ehg \cdot ijp \cdot lmn \cdot qwv \cdot rus,$$

$$s_1^2 s_2^2 s_1 s_2 = ad \cdot bc \cdot eh \cdot fy \cdot ik \cdot jp \cdot ln \cdot mo \cdot qu \cdot rv,$$

$$s_2^2 s_1 s_2 s_1^2 = ae \cdot dh \cdot io \cdot jp \cdot km \cdot ln \cdot qv \cdot ru \cdot st \cdot wx,$$

$$s_2 s_1^2 s_2^2 s_1 = ac \cdot bd \cdot eg \cdot fh \cdot ij \cdot kp \cdot lo \cdot mn \cdot rv \cdot sw,$$

$$s_1 s_2 s_1^2 s_2^2 = ae \cdot cg \cdot ij \cdot kl \cdot mn \cdot op \cdot qx \cdot rs \cdot tu \cdot vw.$$

Having proved the existence of such a group of order 288, we have further to prove that there is only one of this order. This fact follows almost directly from the isomorphism between G and the direct product of two tetrahedral groups. To each of these two tetrahedral groups there corresponds a group of order 24 in G. As this group of order 24 cannot contain a subgroup of order 12 the theorem is proved and we have as a final result the following theorem: *There are four and only four groups that can be generated by two operators of order* 3 *whose commutator is of order* 2. *The orders of these are respectively* 12, 36, 144 *and* 288. The orders of their commutator subgroups are 4, 4, 16 and 32 respectively. As only the last of these is non-abelian, we note that the given group of order 288 is the only group which can be generated by two operators of order 3 whose commutator is of order 2, and which does not have an abelian commutator subgroup.

§ 3. *The groups generated by two operators of orders two and three respectively whose commutator is of order three.*

Let s_1 and s_2 be any two operators which satisfy the following conditions:

$$s_1^2 = 1, \qquad s_2^3 = 1, \qquad (s_1 s_2^2 s_1 s_2)^3 = 1.$$

The existence of such operators is proved by the fact that two generators of the symmetric group of order 6 satisfy these conditions. Moreover, the two substitutions ab, bcd may be substituted for s_1 and s_2 respectively since $ab \cdot bdc \cdot ab \cdot bcd = abc$. Hence it follows that each of the two smallest symmetric groups may be generated by two substitutions which satisfy the conditions imposed on s_1 and s_2. It will soon appear that these are the only symmetric groups which can be generated by two substitutions satisfying the given conditions. In fact, all such groups will be proved to be solvable and to constitute only a small class of the solvable groups.

The three conjugates of $s_1 s_2^2 s_1 s_2$ with respect to s_2 are

$$s_1 s_2^2 s_1 s_2, \qquad s_2^2 s_1 s_2^2 s_1 s_2^2, \qquad s_2 s_1 s_2^2 s_1.$$

Their continued product, in the given order, is the identity. Since the transforms with respect to s_1 of the first and third of these conjugates are respectively the inverses of the operators, it follows that the group generated by any two of them is invariant under the group (G) generated by s_1 and s_2. The group generated by these conjugate operators is the commutator subgroup of G since the conjugates of the commutator of s_1 and s_2 under s_1, s_2 must generate the commutator subgroup of G. We have then as a first result that *the commutator subgroup of G is generated by two operators of order 3 whose product is also of this order.*

The groups which can be generated by two operators of order three whose product is of order three have been studied and it has been proved that the commutator subgroup of such a group is always abelian, and it is either cyclic or it may be generated by two independent operators.* Moreover the quotient group of such a group with respect to the commutator subgroup (i. e., the commutator quotient group) is either of order 3 or of order 9. Hence G is one of those groups in which one arrives at the identity by forming the successive commutator subgroups, or the successive derivatives according to LIE's notation. This proves that G is solvable.†

It will be convenient to represent the commutator subgroup of G by K and the commutator subgroup of K by K_1. It has been observed that K_1 is abelian, that the order of K is either 3 times or 9 times that of K_1 and that K is generated by two operators $(s_2 s_1 s_2^2 s_1,\ s_1 s_2^2 s_1 s_2)$ of order 3 whose product is of this order. It is of interest to observe that some of the groups which satisfy the last condition cannot be used for K in constructing the groups under consideration. Suppose K_1 is cyclic and hence has an abelian group of isomorphisms. The totality of the operators of G which are commutative with a generator of K_1 constitute an invariant subgroup of G. As the corresponding quotient group is abelian this invariant subgroup includes K, and hence the order of K_1 cannot exceed 3. This proves that the order of K is 3^α, $\alpha < 4$, whenever K_1 is cyclic, and that K cannot be any arbitrary group which may be generated by two operators of order 3 whose product is of order 3. ‡

The quotient group of G with respect to K_1 is generated by two operators of orders 2 and 3 respectively whose commutator is of order 3. The order of this quotient group could not be 18, since the commutator subgroup of the group of order 18 which is generated by two such operators is of order 3, and hence this

*Annals of Mathematics, vol. 3 (1901), p. 40.

†American Journal of Mathematics, vol. 20 (1898), p. 277.

‡Annals of Mathematics, loc. cit.

commutator subgroup could not correspond to K. That is, *when K_1 is cyclic its order is* 3 *and the order of G is* 162. The existence of this G is proved by the following substitutions:

$$s_1 = af\cdot bd\cdot ce\cdot jn\cdot ko\cdot lm\cdot x\alpha\cdot yv\cdot wz,$$

$$s_2 = ajs\cdot bkt\cdot clu\cdot dmv\cdot enw\cdot fox\cdot gpy\cdot hqz\cdot ir\alpha,$$

$$s_1 s_2^2 s_1 s_2 = aif\cdot bgd\cdot che\cdot jlk\cdot mno\cdot swz\cdot tx\alpha\cdot uvy,$$

$$s_2 s_1 s_2^2 s_1 = acb\cdot def\cdot jnq\cdot kor\cdot lmp\cdot sax\cdot tyv\cdot uzw.$$

It is not difficult to determine all the groups which may be generated by two operators satisfying the conditions imposed on s_1 and s_2, and which also involve an abelian commutator subgroup. In fact, it follows from the preceding paragraph that the order of K cannot exceed 9 and if this order is 9 the order of G is 54 and G may be represented as a transitive substitution group of degree 9. Moreover, s_1 transforms each operator of K into its inverse since it transforms each of the operators $s_1 s_2^2 s_1 s_2$, $s_2 s_1 s_2^2 s_1$ into its inverse. If such a group exists it must therefore be one of the two simply isomorphic groups of degree 9 known as 54_1 and 54_3.* The existence of this G is proved by the following representations:

$$s_1 = ab\cdot de\cdot gh, \quad s_2 = adg\cdot bfi\cdot ceh, \quad s_1 s_2^2 s_1 s_2 = abc\cdot def, \quad s_2 s_1 s_2^2 s_1 = acb\cdot gih.$$

When the order of K is 3 the order of G is either 6 or 18. In the former case G is evidently the symmetric group of order 6. In the latter case, the group generated by s_2 is not invariant under G since it does not include $s_1 s_2^2 s_1 s_2$. Hence G is the transitive group of degree 6 and order 18; that is, G is the direct product of the symmetric group of order 6 and the group of order 3. We have now proved that *there are three and only three groups which have an abelian commutator subgroup and may be generated by two operators of orders* 2 *and* 3 *respectively whose commutator is of order* 3. The orders of these groups are 6, 18 and 54 respectively.

When the commutator subgroup of G is non-abelian and K_1 is non-cyclic its order cannot be less than 12. When this order is 12 K is the tetrahedral group and the order of G is either 24 or 72. In the former case K includes s_2 and G is the symmetric group of order 24, since this is the only group of this order which has the tetrahedral group for its commutator subgroup.† We have thus found another characteristic property of the symmetric group of order 24 which may be expressed as follows: The octahedral group is the smallest group that can be generated by two operators of orders 2 and 3 respectively having a com-

*Cole, Bulletin of the New York Mathematical Society, vol. 2 (1893), p. 252.
†Quarterly Journal of Mathematics, vol. 28 (1896), p. 282.

mutator of order 3 and has a non-abelian commutator subgroup. When the order of K is 12 and K does not include s_2, G contains the octahedral group generated by s_1 and K as an invariant subgroup. It also contains an invariant subgroup of order 3, since s_2 transforms K in the same manner as one of its own operators. Hence we have proved that *G is either the octahedral group or the direct product of this group and a group of order* 3 *whenever the commutator subgroup of G is of order* 12.

It may be observed from the results of the second paragraph that the commutator quotient group G/K is either of order 2 or the cyclic group of order 6. In the latter case the order of G is divisible by 9 since K does not include s_2. In the former case s_2 is found in K. We proceed to prove that the necessary and sufficient condition that the order of G be divisible by 9 is that its commutator quotient group shall be of order 6. This will be done by proving that we arrive at a contradiction when we assume that the commutator quotient group of G is of order 2 and that the order of G is divisible by 9. When this quotient group is of order 2 the order of K/K_1 cannot be 9, since G/K_1 would then be of order 18 and a group of this order which is generated by an operator of order 3 and an operator of order 2 cannot have a commutator subgroup of order 9. Hence we may assume that the orders of K/K_1 and G/K are 3 and 2 respectively.

All the operators whose orders are not divisible by 3 in K generate an invariant subgroup of G, and the corresponding quotient group is of order 2.3^α. This quotient group is generated by two operators of orders 2 and 3 respectively and its commutator subgroup is of order 3^α. Hence its subgroup of order 3^α cannot be abelian when $\alpha > 1$. If this group of order 3^α were non-abelian its commutator subgroup would be a characteristic subgroup. From this it follows that G must be isomorphic with a group of order 2.3^β, $\beta > 1$, in which the subgroup of order 3^β is abelian, whenever $\alpha > 1$. As this group would be generated by an operator of order 2 and an operator of order 3, and as its commutator subgroup would be of order 3^β, we have arrived at a contradiction; hence it is proved that *the necessary and sufficient condition that the order of G be divisible by* 9 *is that its commutator quotient group shall be of order* 6.

It is not difficult to prove that the number of these groups whose commutator quotient group is of order 2 is infinite. Let p represent one of the infinite system of primes which satisfy the condition $p \equiv 1 \bmod 3$, and consider the holomorph of the group (H) of order p^2 and of type $(1, 1)$. As H contains $p + 1$ subgroups of order p and as its group of isomorphisms (I) contains only $p - 1$ operators which transform each one of these subgroups into itself,* it follows that I has a $(p - 1, 1)$ isomorphism with the triply transitive group of degree $p + 1$ and of order $p(p^2 - 1)$. Each of the operators of the regular cyclic subgroup of order $p - 1$ is transformed into its inverse under this triply

*Transactions of the American Mathematical Society, vol. 2 (1901), p. 260.

transitive group and hence this group includes the symmetric group of order 6.

This symmetric group is simply isomorphic with a subgroup of I, hence the holomorph of H contains a subgroup of order $6p^2$ in which all the subgroups of order 3 are conjugate. To obtain two operators which may be used for s_1 and s_2 we need only select an operator of order 3 and an operator of order 2 which does not transform this operator of order 3 into its inverse. As such operators always exist, it follows that *there is an infinite system of groups which are generated respectively by two operators of orders* 2 *and* 3 *having a commutator of order* 3.

§ 4. *Conclusion.*

The preceding results are analogous to those which relate to the groups of genus zero. As the latter are defined by the orders of two generators and the order of their product and on account of this simple definition are extremely useful, it seems desirable to inquire into those groups which are either completely defined by the orders of two operators and the order of their commutator or are almost completely determined by these properties. The main results which have been obtained may be stated as follows: *Any dihedral groups whose order is divisible by* 8 *may be defined by the orders of two generators and the order of the commutator of these generators*, and it appears very probable that no other non-abelian group can be defined in this manner. When the commutator of two generators is of order 2 and the orders of these generators are either 2, 3 or 3, 3 the groups are almost completely determined by these conditions. In the former case there are two possible groups while in the latter case there are four. When two generators are of order 2 and their commutator is of any odd order there are two possible groups.

There is an infinite system of groups such that each group is generated by two operators whose commutator is of order 2 *while the order of one of the generators is a given number which exceeds* 3 *and the order of the other is any given even number* > 0. This general theorem follows directly from the proof given above that there is such an infinite system when the order of one of the generators is 2 while the order of the other is any number larger that 3. There is also an infinite system of groups each of which may be generated by two operators of orders 3 and 4 respectively whose commutator is of order 2. This fact follows directly from the following substitutions which have a commutator of order 2:

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \cdot mno \cdot pqr,$$

$$s_2 = c\alpha f\beta \cdot i\gamma l\delta \cdot o\epsilon r\theta \cdot ah \cdot bg \cdot de \cdot jq \cdot kp \cdot mn.$$

These substitutions can readily be constructed if it is observed that a cycle of order 3 is transformed into a cycle having two common letters but in the reverse

order by any substitution whose commutator with the cycle is of order 2. It is clear that s_1 and s_2 generate a transitive group and that the degree of this transitive group can be increased beyond any given number by increasing the degrees of s_1, s_2 according to the law which is exhibited by the given substitutions. That there is an infinite system of groups such that each one may be generated by two operators of orders 3 and 5 respectively whose commutator is of order 2 may be deduced in a similar manner from the following substitutions:

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \cdot mno,$$

$$s_2 = bdgmj \cdot cfh\beta l \cdot a\alpha ok\gamma \cdot e\delta ine.$$

If it is desired to double the degree of the transitive group generated by s_1, s_2 the cycle $a\alpha ok\gamma$ may be divided into the two parts $a\alpha \cdots$, $ok\gamma \cdots$ and one of these cycles may be extended by a letter from an additional cycle of s_1 while the remaining four letters necessary to complete these cycles do not belong to s_1. The rest of s_2 may be constructed as before. In this manner we may evidently increase the degree of $\{s_1, s_2\}$ without limit and the commutator of s_1, s_2 will always be of order 2.

URBANA, ILLINOIS.

PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(SECOND MEMOIR)

BY

E. J. WILCZYNSKI

§ 1. *Introduction.*

In the first memoir it was shown † that the projective differential geometry of a surface may be based upon a system of two linear partial differential equations of the second order, which may be reduced to the canonical form

$$\begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned} \tag{1}$$

the coefficients of which are seminvariants; the curves $u = \text{const.}$ and $v = \text{const.}$ will then be asymptotic lines of its integral surface S_y. The integrability conditions of system (1)

$$\begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0, \end{aligned} \tag{2}$$

are supposed to be satisfied, so that (1) has just four linearly independent solutions, y', y'', $y^{(3)}$, $y^{(4)}$, which are interpreted as the homogeneous coordinates of a point P_y of the surface S_y. We shall usually denote the surface S_y by S whenever there is no chance for confusion arising from the suppression of the index.

The semi-covariants of (1) become

$$y = y,\ z = y_u,\ \rho = y_v,\ \sigma = y_{uv}. \tag{3}$$

If four linearly independent solutions y', y'', $y^{(3)}$, $y^{(4)}$, of (1) be put for y into these expressions, there will be found three points P_z, P_ρ, P_σ semi-covariantly connected with the point P_y. These three points, together with P_y, form a

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† These Transactions, vol. 8 (1907), pp. 233–260.

non-degenerate tetrahedron, except for certain singular points. For, if they were coplanar $y', \cdots, y^{(4)}$ would satisfy an equation of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0,$$

which could have at most three linearly independent solutions in common with (1).

We shall make use of this fact very frequently in the following considerations for the purpose of making a detailed study of the surface in the vicinity of a point P_y. In fact we shall use the tetrahedron $P_y P_z P_\rho P_\sigma$ as a tetrahedron of reference, determining the coordinates of any point with respect to it in the following way. An expression of the form

$$\lambda = \alpha y + \beta z + \gamma \rho + \delta \sigma$$

assumes four values $\lambda', \cdots, \lambda^{(4)}$ corresponding to the four values of y. The coordinates of the point P_λ referred to the tetrahedron of reference $P_y P_z P_\rho P_\sigma$ shall be defined by the equations

$$x_1 = \alpha, \ x_2 = \beta, \ x_3 = \gamma, \ x_4 = \delta.$$

§ 2. *The two congruences of asymptotic tangents and their ruled surfaces.*

At each point P_y of the surface there are two asymptotic tangents, which join P_y to P_z and P_ρ respectively. Thus there are two congruences both of which have S as focal surface, the two families of asymptotic lines on S being the cuspidal edges of the developables of these two congruences. We shall speak of one of the two asymptotic tangents through P_y as being of the *first* or *second kind* according as it contains P_z or P_ρ. The corresponding congruences may be distinguished in a like manner.

Let us consider the congruence of asymptotic tangents of the first kind. The line $P_y P_z$ is one of its lines and, as u and v assume all of their values, this line describes the congruence. If u and v be chosen as functions of a single variable t,

$$u = \phi(t), \qquad v = \psi(t),$$

the line $P_y P_z$ will describe a ruled surface of this congruence, whose differential equations may be set up without great difficulty. For our purpose it suffices however to consider certain special ruled surfaces of this congruence. Let $u = u_0$ be the particular asymptotic curve of the second kind which passes through the point P_y. Let a point P move along this curve and construct the asymptotic tangent of the first kind of the point P in each of its positions. The ruled surface thus obtained, generated by $P_y P_z$ as u remains constant, while v assumes all of its values, shall be called *the osculating ruled surface of*

the first kind. It is the locus of the asymptotic tangents of the first kind along an asymptotic curve of the second kind. We shall denote it by R_1. The osculating ruled surface of the second kind R_2 is generated by $P_y P_\rho$ as v remains constant while u assumes all of its values.

We proceed to find the differential equations which characterize R_1. From equations (1) and (3) we find by differentiation

$$\begin{aligned} &y_v = \rho, \qquad y_{vv} = -gy - 2a'z, \\ &z_v = \sigma, \qquad z_{vv} = (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho, \end{aligned} \tag{4}$$

whence

$$\begin{aligned} &y_{vv} + gy + 2a'z = 0, \\ &z_{vv} - 4a'by_v + (g_u - 2a'f)y + (g + 2a'_u)z = 0, \end{aligned} \tag{5}$$

the required equations.

These equations are of the form

$$\begin{aligned} &y_{vv} + p_{11}y_v + p_{12}z_v + q_{11}y + q_{12}z = 0, \\ &z_{vv} + p_{21}y_v + p_{22}z_v + q_{21}y + q_{22}z = 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &p_{11} = 0, \quad p_{12} = 0, \; q_{11} = g, \quad q_{12} = 2a', \\ &p_{21} = -4a'b, \; p_{22} = 0, \; q_{21} = g_u - 2a'f, \; q_{22} = g + 2a'_u, \end{aligned} \tag{7}$$

and therefore serve to characterize the ruled surface R_1 generated by $P_y P_z$ when v alone varies.* The equation $p_{12} = 0$ shows that the curve C_y (the locus of P_y as v assumes its values), which is an asymptotic curve on the surface S, is also an asymptotic curve on R_1.† The curve C_z is a curve on R_1 whose exact geometrical significance will become apparent later.

The invariants of the ruled surface R_1 are expressible in terms of the coefficients (7). One of them, which will be needed in a later paragraph, is

$$\theta = (u_{11} - u_{22})^2 + 4u_{12}u_{21} \tag{8}$$

where

$$\begin{aligned} &u_{11} = -4g, \qquad u_{12} = -8a', \qquad u_{22} = -4g - 8a'_u \\ &u_{21} = -4g_u + 8a'f - 8a'_v b - 8a'b_v = 4a'_{uu} + 8a'f + 8a'b_v. \ddagger \end{aligned} \tag{9}$$

Its vanishing expresses the condition that the two branches of the flecnode curve of R_1 coincide. The equality of the two expressions for u_{21} is a consequence of (2).

* E. J. WILCZYNSKI, *Projective differential geometry of curves and ruled surfaces.* Teubner, Leipzig, 1906, p. 130. We shall hereafter refer to this work as Proj. Diff. Geom.

† Proj. Diff. Geom., p. 142.

‡ Cf. Prof. Diff. Geom., p. 96, eq. (20) for definition of the quantities u_{ik}, and p. 110, eq. (75) for the invariant θ, there denoted by θ_4. Cf. p. 150 for the geometrical significance of θ_4.

The fundamental semi-covariants of (6) are

$$\begin{aligned} \mathfrak{r} &= 2y_v + p_{11}y + p_{12}z = 2\rho, \\ \mathfrak{s} &= 2z_v + p_{21}y + p_{22}z = -4a'by + 2\sigma. \end{aligned} \tag{10}$$

The lines $P_yP_\mathfrak{r}$ and $P_yP_\mathfrak{s}$ are asymptotic tangents to the ruled surface R_1 along its generator P_yP_z. The asymptotic tangent to R_1 at the point $\alpha y + \beta z$ of P_yP_z joins this point to $\alpha\mathfrak{r} + \beta\mathfrak{s}$.† The homogeneous coördinates of an arbitrary point on one of these asymptotic tangents, i. e., the coördinates of an arbitrary point on the hyperboloid H_1, which osculates R_1 along P_yP_z, will therefore be

$$\lambda(\alpha y + \beta z) + \mu(\alpha\mathfrak{r} + \beta\mathfrak{s}) = (\alpha\lambda - 4a'b\mu\beta)y + \lambda\beta z + 2\mu\alpha\rho + 2\mu\beta\sigma.$$

If we introduce the system of coördinates referred to in the introduction, which has $P_yP_vP_\rho P_\sigma$ as its fundamental tetrahedron, we find

$$x_1 = \alpha\lambda - 4a'b\beta\mu, \quad x_2 = \beta\lambda, \quad x_3 = 2\alpha\mu, \quad x_4 = 2\beta\mu,$$

as the coördinates of an arbitrary point on H_1. The equation of H_1 with respect to this tetrahedron of reference is therefore,

$$x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0. \tag{11}$$

The osculating ruled surface of the second kind R_2 is characterized by the equations

$$\begin{aligned} y_{uu} + p'_{11}y_u + p'_{12}\rho_u + q'_{11}y + q'_{12}\rho &= 0, \\ \rho_{uu} + p'_{21}y_u + p'_{22}\rho_u + q'_{21}y + q'_{22}\rho &= 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &p'_{11} = 0, \quad p'_{12} = 0, \ q'_{11} = f, \quad q'_{12} = 2b, \\ &p'_{21} = -4a'b, \ p'_{22} = 0, \ q'_{21} = f_v - 2bg, \ q'_{22} = f + 2b_v. \end{aligned} \tag{13}$$

The invariant θ' whose vanishing expresses the coincidence of the two branches of the flecnode curve of R_2, is

$$\theta' = (u'_{11} - u'_{22})^2 + 4u'_{12}u'_{21}, \tag{14}$$

where

$$\begin{aligned} &u'_{11} = -4f, \quad u'_{12} = -8b, \quad u'_{22} = -4f - 8b_v, \\ &u'_{21} = -4f_v + 8bg - 8a'_vb - 8a'b_v = 4b_{vv} + 8bg + 8ba'_v. \end{aligned} \tag{15}$$

The fundamental semi-covariants of R_2 are

$$\mathfrak{r}' = 2z, \quad \mathfrak{s}' = -4a'by + 2\sigma = \mathfrak{s}. \tag{16}$$

* Proj. Diff. Geom., p. 124. The quantities $\mathfrak{r}$ and $\mathfrak{s}$ are there denoted by ρ and σ.

† For a proof of this statement cf. Proj. Diff. Geom., p. 146.

The equation of the hyperboloid H_2 which osculates R_2 along $P_y P_\rho$ is found to be (11). Therefore: *the same hyperboloid which osculates the osculating ruled surface of the first kind also osculates the osculating ruled surface of the second kind. The two sets of generators on this hyperboloid are the asymptotic tangents of the two osculating ruled surfaces.*

§ 3. *The osculating linear complexes of the osculating ruled surfaces.*

Consider the ruled surface R_1, which is determined by the system of equations (6). We proceed to determine its osculating linear complex. According to the general theory of ruled surfaces, this may be done as follows.* Let θ be different from zero. Then the flecnode curve of R_1 has two distinct branches C_η and C_ζ, where

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{17}$$

and where δ/β and γ/α are the two (distinct) roots of the quadratic

$$u_{12}\lambda^2 + (u_{11} - u_{22})\lambda - u_{21} = 0. \tag{18}$$

Perform the transformation (17), (18). The surface R_1 is then referred to its flecnode curve. Let the result of this transformation upon (6) be the system

$$\begin{aligned} \eta_{vv} + \pi_{11}\eta_v + \pi_{12}\zeta_v + \kappa_{11}\eta + \kappa_{12}\zeta &= 0, \\ \zeta_{vv} + \pi_{21}\eta_v + \pi_{22}\zeta_v + \kappa_{21}\eta + \kappa_{22}\zeta &= 0. \end{aligned} \tag{19}$$

Let r and s be the fundamental semi-covariants of this system, i. e., let

$$r = 2\eta_v + \pi_{11}\eta + \pi_{12}\zeta, \qquad s = 2\zeta_v + \pi_{21}\eta + \pi_{22}\zeta. \tag{20}$$

Referred to the tetrahedron $P_\eta P_\zeta P_r P_s$, the equation of the osculating linear complex will be

$$\pi_{12}\bar{\omega}_{13} + \pi_{21}\bar{\omega}_{42} = 0, \tag{21}$$

$\bar{\omega}_{ik}$ being the line coordinates with reference to this special tetrahedron.†

Substituting the values (9) in (18) we find that we may take in (17)

$$\begin{aligned} \alpha &= 16a', \qquad \beta = 16a', \\ \gamma &= 8a'_u + \sqrt{\theta}, \quad \delta = 8a'_u - \sqrt{\theta}, \end{aligned} \tag{22}$$

whence

$$\Delta = \alpha\delta - \beta\gamma = -32a'\sqrt{\theta}. \tag{23}$$

* Proj. Diff. Geom., p. 150. Another proof, independent of the general theory of ruled surfaces will also be given.

† Proj. Diff. Geom., p. 206.

We find, consequently, either by direct substitution or by making use of the general formulæ of the theory of ruled surfaces,*

$$
\begin{aligned}
\pi_{11} &= \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta}, \\
\pi_{12} &= \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta}, \\
\pi_{21} &= \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta}, \\
\pi_{22} &= \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},
\end{aligned} \tag{24}
$$

where

$$
C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b. \tag{25}
$$

It remains to find the relation between the original tetrahedron $P_y P_z P_\rho P_\sigma$ and the new one $P_\eta P_\zeta P_r P_s$.

We have, in the first place, from (17) and (22),

$$
\begin{aligned}
y &= 16a'\eta + 16a'\zeta, \\
z &= B\eta + A\zeta,
\end{aligned} \tag{26}
$$

where we have put

$$
A = 8a'_u - \sqrt{\theta}, \qquad B = 8a'_u + \sqrt{\theta}, \tag{27}
$$

for abbreviation. Therefore

$$
\begin{aligned}
-32a'\sqrt{\theta}\eta &= Ay - 16a'z, \\
-32a'\sqrt{\theta}\zeta &= -By + 16a'z.
\end{aligned}
$$

It is known moreover that a transformation of form (17) gives rise to a cogredient transformation of the fundamental semi-covariants $\mathfrak{r}$, $\mathfrak{s}$ into r, s.† We shall, therefore, have

$$
\begin{aligned}
-32a'\sqrt{\theta}r &= A\mathfrak{r} - 16a'\mathfrak{s}, \\
-32a'\sqrt{\theta}s &= -B\mathfrak{r} + 16a'\mathfrak{s}.
\end{aligned}
$$

Consequently we find

$$
\begin{aligned}
-32a'\sqrt{\theta}\eta &= Ay - 16a'z, \\
-32a'\sqrt{\theta}\zeta &= -By + 16a'z, \\
-32a'\sqrt{\theta}r &= 64a'^2 by + 2A\rho - 32a'\sigma, \\
-32a'\sqrt{\theta}s &= -64a'^2 by - 2B\rho + 32a'\sigma.
\end{aligned} \tag{28}
$$

* Proj. Diff. Geom., p. 102.

† Proj. Diff. Geom., p. 124.

Let $x_1, \cdots, x_4$ be the coördinates of a point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ and let $\bar{x}_1, \cdots, \bar{x}_4$ be the coördinates of the same point with respect to the tetrahedron $P_\eta P_\zeta P_r P_s$. Then, the two expressions

$$x_1 y + x_2 z + x_3 \rho + x_4 \sigma \qquad \text{and} \qquad \bar{x}_1 \eta + \bar{x}_2 \zeta + \bar{x}_3 r + \bar{x}_4 s$$

can differ only by a factor. Let ω be a proportionality factor. We shall have

$$\begin{aligned} \omega x_1 &= A\bar{x}_1 - B\bar{x}_2 + 64a'^2 b\bar{x}_3 - 64a'^2 b\bar{x}_4, \\ \omega x_2 &= -16a'\bar{x}_1 + 16a'\bar{x}_2, \\ \omega x_3 &= 2A\bar{x}_3 - 2B\bar{x}_4, \\ \omega x_4 &= -32a'\bar{x}_3 + 32a'\bar{x}_4, \end{aligned} \tag{29}$$

whence, if ω' denotes another factor of proportionality,

$$\begin{aligned} \omega'\bar{x}_1 &= 32a'x_1 + 2Bx_2 + 64a'^2 bx_4, \\ \omega'\bar{x}_2 &= 32a'x_1 + 2Ax_2 + 64a'^2 bx_4, \\ \omega'\bar{x}_3 &= 16a'x_3 + Bx_4, \\ \omega'\bar{x}_4 &= 16a'x_3 + Ax_4. \end{aligned} \tag{30}$$

Let y_i, z_i be the coördinates of two points on a line l, and let $\bar{y}_i, \bar{z}_i$ be the coördinates of the same two points referred to the tetrahedron $P_\eta P_\zeta P_r P_s$. Let the Plückerian coördinates of the line with respect to the two systems be

$$\omega_{ik} = y_i z_k - y_k z_i \qquad \text{and} \qquad \bar{\omega}_{ik} = \bar{y}_i \bar{z}_k - \bar{y}_k \bar{z}_i$$

respectively. We shall have (omitting proportionality factors),

$$\begin{aligned} \bar{\omega}_{13} &= 16^2 a'^2 \omega_{13} + 16Ba'\omega_{23} - 2\cdot 16^2 a'^3 b\omega_{34} + 16Ba'\omega_{14} - B^2\omega_{42}, \\ \bar{\omega}_{42} &= -16^2 a'^2 \omega_{13} - 16Aa'\omega_{23} + 2\cdot 16^2 a'^3 b\omega_{34} - 16Aa'\omega_{14} + A^2\omega_{42}. \end{aligned} \tag{31}$$

It only remains to substitute these expressions and the expressions (24) for π_{12} and π_{21} into (21) and to simplify. *The equation of the linear complex C_1, which osculates the osculating ruled surface of the first kind at the point P, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, will thus be found to be*

$$a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{31}\omega_{34} + a_{42}\omega_{42} = 0 \tag{32a}$$

where

$$\begin{aligned} a_{12} &= 0, \qquad a_{13} = 2^8 a'^2 C, \qquad a_{14} = -2^3(a'\theta_v - 2a'_v\theta - 2^4 a' a'_u C), \\ a_{23} &= a_{14}, \qquad a_{34} = -2^9 a'^3 bC, \\ a_{42} &= -\left[C(\theta + 64a'^2_u) + 16\theta\frac{a'_u a'_v}{a'} - 8a'_u\theta_v \right]. \end{aligned} \tag{32b}$$

The invariant of this complex is

$$(33) \qquad \mathfrak{A} = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 2^6\left[(a'\theta_v - 2a'_v\theta)^2 - 4a'^2C^2\theta\right].$$

It may be noted that, although our investigation was made under the hypothesis $\theta \neq 0$, the equations (32) will be valid also if $\theta = 0$.* If $a' = 0$, however, not only do these equations become meaningless but the complex C_1 itself becomes indeterminate. In fact, in that case the original surface S is itself a ruled surface; it coincides with R_2, and the surface R_1 is the hyperboloid which osculates R_2 along that one of its generators which passes through P. Now a hyperboloid belongs to a double infinity of linear complexes, so that the osculating linear complex is necessarily indeterminate.

More generally we may say that, *whenever the complex C_1 is indeterminate, five consecutive generators of R_1 have a pair of straight line intersectors.* Excluding the case $a' = 0$ which we have already discussed, equations (32) show that C_1 *is indeterminate if*

$$(34) \qquad C = 0, \qquad a'\theta_v - 2a'_v\theta = 0.$$

Moreover, if the conditions (34) are satisfied identically the osculating ruled surface of S belongs to a linear congruence. This follows from the fact that if equations (34) are satisfied, and if $\theta \neq 0$, we find from (24) $\pi_{12} = \pi_{21} = 0$, so that C_η and C_ζ are at the same time flecnode curves and asymptotic curves on R_1; but this is possible only if they are straight lines.

If $\theta = 0$, $C = 0$, R_1 belongs to a linear congruence with coincident directrices. In that case, in fact, we make the substitution

$$y = \eta, \qquad z = \frac{a'_u}{2a'}\eta - \zeta,$$

in place of (17). We find a new system of form (6) for η and ζ whose coefficients π_{ik} are zero while the quantities $\bar{u}_{ik}$ corresponding to (9) become

$$\bar{u}_{11} - \bar{u}_{22} = 0, \qquad \bar{u}_{12} = 8a, \qquad \bar{u}_{21} = 0,$$

and this is sufficient to prove the above statement.†

If the conditions (34) *are not satisfied identically but only for certain values of u and v, five consecutive generators of the osculating ruled surface of the first kind which belongs to that point (u, v) of S_1 have a pair of straight line intersectors. These are distinct or not according as θ is or is not different from zero.*

If $\mathfrak{A} = 0$ the complex C_1 is special. Equation (33) therefore gives rise to the following theorem: *if the two branches of the flecnode curve of the oscu-*

* That (32) gives the osculating linear complex also in that case will be proved presently.

† Proj. Diff. Geom. p. 170.

lating ruled surface of the first kind coincide, its osculating linear complex is special.

It remains to prove equations (32) by another method, independent of the general theory of ruled surfaces, and not restricted by the assumption $\theta \neq 0$. In order to do this we shall develop the Plückerian coördinates of the line $P_y P_z$ according to powers of the increment δv, calculating the terms up to and including the fourth order. The elimination of the powers of δv will then give the equation of the osculating linear complex.

We have

$$y = y, \qquad y_v = \rho, \qquad y_{vv} = -gy - 2a'z,$$

$$y_{vvv} = -g_v y - 2a'_v z - g\rho - 2a'\sigma, \tag{35}$$

$$y_{vvvv} = \epsilon' y + \epsilon'' z + \epsilon^{(3)}\rho + \epsilon^{(4)}\sigma,$$

where

$$\begin{aligned} \epsilon' &= -g_{vv} + g^2 - 4a'^2 f + 2a' g_u, \\ \epsilon'' &= 4a' a'_u + 4a' g - 2a'_{vv}, \\ \epsilon^{(3)} &= -2g_v - 8a'^2 b, \\ \epsilon^{(4)} &= -4a'_v. \end{aligned} \tag{36}$$

These equations may be obtained from (1) by differentiation, remembering that

$$y_u = z, \qquad y_v = \rho, \qquad y_{uv} = \sigma.$$

Similarly, we find

$$z = z, \qquad z_v = \sigma, \qquad z_{vv} = (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho,$$

$$z_{vvv} = \delta' y + \delta'' z + \delta^{(3)}\rho + \delta^{(4)}\sigma, \tag{37}$$

$$z_{vvvv} = \zeta' y + \zeta'' z + \zeta^{(3)}\rho + \zeta^{(4)}\sigma,$$

where

$$\begin{aligned} \delta' &= -g_{uv} + 2a'_v f + 2a' f_v - 4a'bg, \\ \delta'' &= -g_v - 2a'_{uv} - 8a'^2 b, \\ \delta^{(3)} &= -g_u + 4a'_v b + 4a' b_v + 2a' f, \\ \delta^{(4)} &= -g - 2a'_u, \end{aligned} \tag{38}$$

and

$$\begin{aligned} \zeta' &= \delta'_v - \delta^{(3)} g + \delta^{(4)}(2a'f - g_u), \\ \zeta'' &= \delta''_v - 2a'\delta^{(3)} - \delta^{(4)}(g + 2a'_u), \\ \zeta^{(3)} &= \delta^{(3)}_v + \delta' + 4\delta^{(4)} a' b \\ \zeta^{(4)} &= \delta^{(4)}_v + \delta''. \end{aligned} \tag{39}$$

We shall have for a point Y, in the vicinity of the point (u, v) of the surface S, if u remains constant, while v alone varies,

$$Y = y + y_v \delta v + \tfrac{1}{2} y_{vv} (\delta v)^2 + \tfrac{1}{6} y_{vvv} (\delta v)^3 + \tfrac{1}{24} y_{vvvv} (\delta v)^4 + \cdots,$$

which may be written in the form

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma,$$

where $y_1, \cdots, y_4$ will be the homogeneous coördinates of the point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, and where $y_1, \cdots, y_4$ will be power-series proceeding according to powers of δv. Similarly we denote by $z_1, \cdots, z_4$ the coördinates of the point Z. We find in this way, putting $\delta v = t$ for brevity:

$$\begin{aligned}
y_1 &= 1 - \tfrac{1}{2} g t^2 - \tfrac{1}{6} g_v t^3 + \tfrac{1}{24} \epsilon' t^4 + \cdots, \\
y_2 &= -a' t^2 - \tfrac{1}{3} a'_v t^3 + \tfrac{1}{24} \epsilon'' t^4 + \cdots, \\
y_3 &= t - \tfrac{1}{6} g t^3 + \tfrac{1}{24} \epsilon^{(3)} t^4 + \cdots, \\
y_4 &= -\tfrac{1}{3} a' t^3 + \tfrac{1}{24} \epsilon^{(4)} t^4 + \cdots,
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
z_1 &= \tfrac{1}{2}(2a'f - g_u) t^2 + \tfrac{1}{6} \delta' t^3 + \tfrac{1}{24} \zeta' t^4 + \cdots, \\
z_2 &= 1 - \tfrac{1}{2}(g + 2a'_u) t^2 + \tfrac{1}{6} \delta'' t^3 + \tfrac{1}{24} \zeta'' t^4 + \cdots, \\
z_3 &= 2a'b t^2 + \tfrac{1}{6} \delta^{(3)} t^3 + \tfrac{1}{24} \zeta^{(3)} t^4 + \cdots, \\
z_4 &= t + \tfrac{1}{6} \delta^{(4)} t^3 + \tfrac{1}{24} \zeta^{(4)} t^4 + \cdots.
\end{aligned} \tag{41}$$

The Plückerian coördinates of the line joining the points (y_i) and (z_i) are

$$\omega_{ik} = y_i z_k - y_k z_i.$$

Consequently we find

$$\begin{aligned}
\omega_{13} &= 2a'b t^2 + \tfrac{1}{6}[\delta^{(3)} - 3(2a'f - g_u)] t^3 + \tfrac{1}{24}[\zeta^{(3)} - 4\delta' - 24a'bg] t^4 + \cdots, \\
\omega_{14} &= t + \tfrac{1}{6}(\delta^{(4)} - 3g) t^3 + \tfrac{1}{24}(\zeta^{(4)} - 4g_v) t^4 + \cdots, \\
\omega_{23} &= -t + \tfrac{1}{6}(4g + 6a'_u) t^3 - \tfrac{1}{24}(\epsilon^{(3)} + 4\delta'' + 48a'^2 b) t^4 + \cdots, \\
\omega_{34} &= t^2 + \tfrac{1}{6}(\delta^{(4)} - g) t^4 + \cdots, \\
\omega_{42} &= \tfrac{2}{3} a' t^3 + \tfrac{1}{24}(\epsilon^{(4)} + 8a'_v) t^4 + \cdots,
\end{aligned} \tag{42}$$

whence

$$\begin{aligned}
\omega_{14} + \omega_{23} &= \tfrac{1}{6}(\delta^{(4)} + g + 6a'_u) t^3 + \tfrac{1}{24}(\zeta^{(4)} - \epsilon^{(3)} - 4\delta'' - 4g_v - 48a'^2 b) t^4 + \cdots, \\
\omega_{13} - 2a'b\omega_{34} &= \tfrac{1}{6}(\delta^{(3)} - 6a'f + 3g_u) t^3 + \tfrac{1}{24}(\zeta^{(3)} - 4\delta' - 16a'bg - 8a'b\delta^{(4)}) t^4 + \cdots, \\
\omega_{42} &= \tfrac{2}{3} a' t^3 + \tfrac{1}{24}(\epsilon^{(4)} + 8a'_v) t^4 + \cdots.
\end{aligned}$$

These equations may be written more simply:

$$\begin{aligned} \omega_{14} + \omega_{23} &= lt^3 + mt^4 + \cdots, \\ \omega_{13} - 2a'b\omega_{34} &= nt^3 + pt^4 + \cdots, \\ \omega_{42} &= qt^3 + rt^4 + \cdots, \end{aligned} \tag{43}$$

where

$$\begin{aligned} l &= \tfrac{2}{3}a'_u, \qquad m = \tfrac{1}{6}(a'_{uv} - 4a'^2b), \\ n &= \tfrac{1}{6}(2g_u + 4a'_vb + 4a'b_v - 4a'f) = -\tfrac{1}{12}u_{21}, \\ p &= \tfrac{1}{12}(g_{uv} + 2a'_{vv}b + 4a'_vb_v + 2a'b_{vv} - 2a'_uf - 2a'f_v + 4a'ba'_u) \\ &= \tfrac{1}{12}\left(-\tfrac{1}{4}\frac{\partial u_{21}}{\partial v} + 4a'ba'_u\right), \\ q &= \tfrac{2}{3}a', \qquad r = \tfrac{1}{6}a'_v. \end{aligned} \tag{44}$$

If t^3 and t^4 are eliminated from (43), the equation of the osculating linear complex is obtained in the form

$$(pq - nr)(\omega_{14} + \omega_{23}) + (lr - mq)(\omega_{13} - 2a'b\omega_{34}) + (mn - lp)\omega_{42} = 0,$$

and this becomes identical with (32) since

$$pq - nr = \frac{a_{14}}{-2^{11}\cdot 3^2a'}, \qquad lr - mq = \frac{a_{13}}{-2^{11}\cdot 3^2a'}, \qquad mn - lp = \frac{a_{42}}{-2^{11}\cdot 3^2a'}.$$

In the same way we may find the equation of the linear complex which osculates the osculating ruled surface of the second kind, R_2. It is

$$b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0 \tag{45a}$$

where

$$\begin{aligned} &b_{12} = 2^8b^2C', \ b_{13} = 0, \ b_{14} = -b_{23} = -2^3(b\theta'_u - 2b_u\theta' - 2^4bb_vC'), \\ &b_{34} = C'(\theta' + 64b_v^2) + 16\theta'\frac{b_ub_v}{b} - 8b_v\theta'_u, \ b_{42} = 2^9a'b^3C', \end{aligned} \tag{45b}$$

the quantities C' and θ' being defined by the equations

$$\begin{aligned} C' &= 8b_{uv} - 8\frac{b_ub_v}{b} - 32a'b^2, \\ \theta' &= (u'_{11} - u'_{22})^2 + 4u'_{12}u'_{21}, \end{aligned} \tag{46}$$

where u'_{ik} are the quantities (15).

We shall speak of this complex as the complex C_2. *Its invariant is*

$$\mathfrak{B} = -2^6[(b\theta'_u - 2b_u\theta')^2 - 4b^2C'^2\theta'\,]. \tag{47}$$

The simultaneous invariant of the two complexes C_1 and C_2 is

$$(\mathfrak{A}, \mathfrak{B}) = 0; \tag{48}$$

i. e., *the linear complexes C_1 and C_2, which osculate the two osculating ruled surfaces, are in involution.* The congruence common to the two complexes has distinct directrices unless one of them is special. If both are special, their axes intersect.

A linear complex always determines a null-system, i. e., a point-plane correspondence with corresponding elements in united position. Let P' be a point on the generator g_1 of R_1 which passes through P. The plane which corresponds to it in the linear complex C_1 contains g_1; that which corresponds to it in C_2 contains the tangent of the asymptotic curve of R_1 which passes through that point. This may be verified without any difficulty by setting up the equations of the null-systems concerned. In general, a linear complex associates with each point of a surface a direction through that point, namely the intersection of its tangent plane with its null-plane. Let us, with LIE, speak of the configuration of a point with a line through it, as a *line-element.* The ∞^2 line-elements which a linear complex determines on a surface may be collected so as to give rise to a single infinity of curves. Thus there is upon every surface a single infinity of curves whose tangents belong to a given linear complex. These may be called the *complex-curves* of the surface. We extend this idea of complex-curve by considering, in connection with every generator of R_1, a different linear complex, namely the complex C_2 which is associated with that generator. We may then express our above result as follows.

The complex curves, determined upon the osculating ruled surface of one kind by the osculating linear complex of the osculating ruled surface of the other kind, are its asymptotic lines.

§ 4. *The asymptotic lines of the surface and their osculating linear complexes.*

Consider an asymptotic curve Γ' of the first kind. It will be generated by the motion of P_y as v remains constant while u passes through all of its values. The line $P_y P_z$ will generate the developable of which the asymptotic curve is the cuspidal edge.

The linear differential equation of the fourth order which is characteristic of Γ' will be obtained by eliminating z, ρ, σ from the four equations

$$\begin{aligned} y_u &= z, \qquad y_{uu} = -fy - 2b\rho, \\ y_{uuu} &= -f_u y - fz - 2b_u\rho - 2b\sigma, \\ y_{uuuu} &= (f^2 - 4b^2 g + 2bf_v - f_{uu})y - 2(f_u + 4a'b^2)z \\ &\qquad + 2(2bf + 2bb_v - b_{uu})\rho - 4b_u\sigma. \end{aligned} \tag{49}$$

The result is

$$y_{uuuu} + 4p_1' y_{uuu} + 6p_2' y_{uu} + 4p_3' y_u + p_4' y = 0, \tag{50a}$$

where

$$p_1' = -\frac{b_u}{2b}, \qquad p_2' = \tfrac{1}{3}(f + b_v) - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right),$$

$$p_3' = \tfrac{1}{2}\left(f_u + 4a'b^2 - f\frac{b_u}{b}\right), \tag{50b}$$

$$p_4' = f_{uu} - f^2 + 4b^2 g - 2bf_v + \frac{f}{b}(2bf + 2bb_v - b_{uu}) - 2\frac{b_u}{b^2}(bf_u - b_u f).$$

The seminvariants of (50a) may be computed without difficulty.* For our present purpose we need only

$$P_2' = p_2' - \frac{\partial p_1'}{\partial u} - p_1'^2,$$

and this becomes

$$P_2' = \tfrac{1}{3}(f + b_v) + \tfrac{1}{3}\frac{b_{uu}}{b} - \tfrac{5}{12}\frac{b_u^2}{b^2}. \tag{51}$$

We shall denote the fundamental semi-covariants of (50) by y', z', ρ', σ', so that

$$y' = y, \qquad z' = y_u + p_1' y,$$

$$\rho' = y_{uu} + 2p_1' y_u + p_2' y,$$

$$\sigma' = y_{uuu} + 3p_1' y_{uu} + 3p_2' y_u + p_3' y.\dagger$$

Express these in terms of the fundamental semi-covariants of the surface S. We find

$$y' = y, \qquad z' = -\frac{b_u}{2b} y + z,$$

$$\rho' = \left[\tfrac{1}{3}(b_v - 2f') - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right] y - \frac{b_u}{b} z - 2b\rho, \tag{52}$$

$$\sigma' = \left[\tfrac{1}{2}(-f_u + 4a'b^2) + \frac{b_u}{b} f\right] y + \left[b_v - \tfrac{1}{2}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right] z + b_u \rho - 2b\sigma.$$

Consequently, if $x_1, \cdots, x_4$ denote the coördinates of a point with respect to the tetrahedron of reference $P_y P_z P_\rho P_\sigma$, and if $x_1', \cdots, x_4'$ denote the coördinates of the same point referred to the tetrahedron $P_{y'} P_{z'} P_{\rho'} P_{\sigma'}$, we shall have

* Proj. Diff. Geom., p. 239.

† Ibid., p. 239.

$$\omega x_1 = x_1' - \frac{b_u}{2b}x_2' + \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right]x_3' + \left(-\tfrac{1}{2}f_u + 2a'b^2 + \frac{b_u}{b}f\right)x_4',$$

(53)
$$\omega x_2 = x_2' - \frac{b_u}{b}x_3' + \left[b_v - \tfrac{1}{2}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right]x_4',$$

$$\omega x_3 = -2bx_3' + b_u x_4',$$

$$\omega x_4 = -2bx_4',$$

where ω is a proportionality factor.

We find, therefore, if ω' denotes another factor of proportionality,

$$\omega' x_1' = 2bx_1 + b_u x_2 + \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\frac{b_{uu}}{b} - \tfrac{1}{6}\frac{b_u^2}{b^2}\right]x_3 + \left[-\tfrac{1}{2}f_u + 2a'b^2 + \tfrac{1}{3}\frac{b_u}{b}\left(2f + 2b_v - \frac{b_{uu}}{b}\right) + \tfrac{5}{12}\frac{b_u^3}{b^3}\right]x_4,$$

(54)
$$\omega' x_2' = 2bx_2 - \frac{b_u}{b}x_3 + \left(b_v - \tfrac{1}{2}\frac{b_{uu}}{b} + \tfrac{1}{2}\frac{b_u^2}{b^2}\right)x_4,$$

$$\omega' x_3' = -x_3 - \frac{b_u}{2b}x_4,$$

$$\omega' x_4' = -x_4.$$

The equation of the osculating linear complex of Γ' referred to the tetrahedron $P_{y'}P_{z'}P_{\rho'}P_{\sigma'}$ is

(55)
$$\omega_{14}' - 2P_2'\omega_{34}' - \omega_{23}' = 0,$$

where ω_{ik}' are the Plückerian coördinates of a line with respect to the tetrahedron just mentioned.*

Let ω_{ik} denote the Plückerian coördinates of a line with respect to the tetrahedron $P_yP_zP_\rho P_\sigma$. Then we shall find from (54), (omitting proportionality factors),

$$\omega_{14}' = -2b\omega_{14} + b_u\omega_{42} - \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\frac{b_{uu}}{b} - \tfrac{1}{6}\frac{b_u^2}{b^2}\right]\omega_{34},$$

(56)
$$\omega_{34}' = \omega_{34}, \qquad \omega_{23}' = -2b\omega_{23} + \left(b_v - \tfrac{1}{2}\frac{b_{uu}}{b} + \frac{b_u^2}{b^2}\right)\omega_{34} + b_u\omega_{42}.$$

Substitution of these values and of (51) into (55) gives

(57)
$$-b_v\omega_{34} - b\omega_{14} + b\omega_{23} = 0,$$

* Proj. Diff. Geom., p. 253.

as the equation of the linear complex C' which osculates the asymptotic curve Γ' of the first kind. Its invariant is

$$\mathfrak{A}' = -b^2. \tag{58}$$

Therefore, the complex C' can be special only if the asymptotic curve Γ' is a straight line.

The differential equation of the asymptotic curve of the second kind Γ'' which passes through P_y is

$$y_{vvvv} + 4p_1'' y_{vvv} + 6p_2'' y_{vv} + 4p_3'' y_v + p_4'' y = 0, \tag{59a}$$

where

$$p_1'' = -\frac{a'_v}{2a'} \qquad p_2'' = \tfrac{1}{3}(g + a'_u) - \tfrac{1}{6}\left(\frac{a'_{vv}}{a'} - 2\frac{a'^2_v}{a'^2}\right),$$

$$p_3'' = \tfrac{1}{2}(g_v + 4a'^2 b) - \tfrac{1}{2} g \frac{a'_v}{a'}, \tag{59b}$$

$$p_4'' = g_{vv} - g^2 + 4a'^2 f - 2a'g_u + \frac{g}{a'}(2a'g + 2a'a'_u - a'_{vv}) - 2\frac{a'_v}{a'^2}(a'g_v - a'_v g).$$

Its simplest semi-invariant is

$$P_2'' = \tfrac{1}{3}(g + a'_u) + \tfrac{1}{3}\frac{a'_{vv}}{a'} - \tfrac{5}{12}\frac{a'^2_v}{a'^2}. \tag{60}$$

Its fundamental semi-covariants are

$$y'' = y, \qquad z'' = -\frac{a'_v}{2a'} y + \rho,$$

$$\rho'' = \left[\tfrac{1}{3}(a'_u - 2g) - \tfrac{1}{6}\left(\frac{a'_{vv}}{a'} - 2\frac{a'^2_v}{a'^2}\right)\right] y - 2a'z - \frac{a'_v}{a'}\rho, \tag{61}$$

$$\sigma'' = \left[\tfrac{1}{2}(-g_v + 4a'^2 b) + \frac{a'_v}{a'} g\right] y + a'_v z + \left[a'_u - \tfrac{1}{2}\left(\frac{a'_{vv}}{a'} - 2\frac{a'^2_v}{a'^2}\right)\right]\rho - 2a'\sigma.$$

If $x_1, \cdots, x_4$ are the coördinates of a point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ and $x_1'', \cdots, x_4''$ are the coördinates of the same point with reference to the tetrahedron $P_{y''} P_{z''} P_{\rho''} P_{\sigma''}$, we find, therefore,

$$\omega x_1 = x_1'' - \frac{a'_v}{2a'} x_2'' + \left[\tfrac{1}{3}(a'_u - 2g) - \tfrac{1}{6}\left(\frac{a'_{vv}}{a'} - 2\frac{a'^2_v}{a'^2}\right)\right] x_3'' + \left(-\tfrac{1}{2} g_v + 2a'^2 b + \frac{a'_v}{a'} g\right) x_4'',$$

$$\omega x_2 = -2a' x_3'' + a'_v x_4'', \tag{62}$$

$$\omega x_3 = x_2'' - \frac{a'_v}{a'} x_3'' + \left[a'_u - \tfrac{1}{2}\left(\frac{a'_{vv}}{a'} - 2\frac{a'^2_v}{a'^2}\right)\right] x_4'',$$

$$\omega x_4 = -2a' x_4'',$$

where ω is a factor of proportionality. Consequently, if ω'' denotes a second proportionality factor,

$$\omega'' x_1'' = 2a' x_1 + \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\frac{a_{vv}'}{a'} - \tfrac{1}{6}\frac{a_v'^2}{a'^2}\right] x_2 + a_v' x_3$$
$$+ \left[-\tfrac{1}{2}g_v + 2a'^2 b + \tfrac{1}{3}\frac{a_v'}{a'}\left(2g + 2a_u' - \frac{a_{vv}'}{a'}\right) + \tfrac{5}{12}\frac{a_v'^3}{a'^3}\right] x_4,$$

(63)
$$\omega'' x_2'' = -\frac{a_v'}{a'} x_2 + 2a' x_3 + \left(a_u' - \tfrac{1}{2}\frac{a_{vv}'}{a'} + \tfrac{1}{2}\frac{a_v'^2}{a'^2}\right) x_4,$$

$$\omega'' x_3'' = -x_2 - \frac{a_v'}{2a'} x_4,$$

$$\omega'' x_4'' = -x_4.$$

The equation of the osculating linear complex C'' of Γ'' referred to the tetrahedron $P_{y''} P_{z''} P_{\rho''} P_{\sigma''}$ is

$$\omega_{14}'' - 2P_2'' \omega_{34}'' - \omega_{23}'' = 0.$$

We find

$$\omega_{14}'' = -2a'\omega_{14} + \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\frac{a_{vv}'}{a'} - \tfrac{1}{6}\frac{a_v'^2}{a'^2}\right]\omega_{42} - a_v'\omega_{34},$$
$$\omega_{34}'' = -\omega_{42},$$
$$\omega_{23}'' = +2a'\omega_{23} - \left(a_u' - \tfrac{1}{2}\frac{a_{vv}'}{a'} + \tfrac{1}{2}\frac{a_v'^2}{a'^2}\right)\omega_{12} - \tfrac{1}{2}\frac{a_v'^2}{a'^2}\omega_{42},$$

so that the above equation becomes

(64)
$$-a_u'\omega_{42} + a'\omega_{14} + a'\omega_{23} = 0.$$

Equation (64) *characterizes the linear complex C'' which osculates the asymptotic curve Γ'' of the second kind*, the tetrahedron of reference being $P_y P_z P_\rho P_\sigma$. *The invariant of C'' is*

(65)
$$\mathfrak{A}'' = a'^2.$$

The complexes C' and C'' are in involution. In fact their simultaneous invariant vanishes,

(66)
$$(\mathfrak{A}', \mathfrak{A}'') = 0.$$

Therefore, *the osculating linear complexes of the two asymptotic curves which intersect in a point of a surface are always in involution.*

Let

(67)
$$\Sigma a_{ik}'\omega_{ik} = 0, \qquad \Sigma a_{ik}''\omega_{ik} = 0,$$

be two linear complexes. Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be their respective invariants, and let $(\mathfrak{A}', \mathfrak{A}'')$ denote their simultaneous invariant. Then the equations of the

directrices of the congruence which the two complexes have in common may be obtained by writing down the conditions which must be satisfied by a point in order that the same plane may correspond to it in both complexes. We thus find

$$
\begin{aligned}
&* + (a'_{12} - \omega_k a''_{12})x_2 + (a'_{13} - \omega_k a''_{13})x_3 + (a'_{14} - \omega_k a''_{14})x_4 = 0,\\
&(-a'_{12} + \omega_k a''_{12})x_1 + * + (a'_{23} - \omega_k a''_{23})x_3 + (a'_{24} - \omega_k a''_{24})x_4 = 0,\\
&(-a'_{13} + \omega_k a''_{13})x_1 + (-a'_{23} + \omega_k a''_{23})x_2 + * + (a'_{34} - \omega_k a''_{34})x_4 = 0,\\
&(-a'_{14} + \omega_k a''_{14})x_1 + (-a'_{24} + \omega_k a''_{24})x_2 + (-a'_{34} + \omega_k a''_{34})x_3 + * = 0,
\end{aligned}
\tag{68}
$$

where $a'_{24} = -a'_{42}$, $a''_{24} = -a''_{42}$, and where ω_1, ω_2 are the two roots of the quadratic

$$
\mathfrak{A}'\omega^2 - (\mathfrak{A}', \mathfrak{A}'')\omega + \mathfrak{A}'' = 0. \tag{69}
$$

In the present case we may put

$$
\omega_1 = \frac{a'}{b}, \qquad \omega_2 = -\frac{a'}{b},
$$

so that the equations of the two directrices become

$$
x_4 = 0, \qquad 2a'bx_1 + ba'_u x_2 + a'b_v x_3 = 0, \tag{70a}
$$

and

$$
2bx_2 + b_v x_4 = 0, \qquad 2a'x_3 + a'_u x_4 = 0. \tag{70b}
$$

We shall denote them by the letters, d and d', and shall speak of them as being of the first or second kind respectively. From equations (70) we deduce the following result. *One of the directrices of the congruence common to the osculating linear complexes of the two asymptotic curves of a point lies in the tangent plane of this point, while the other one passes through the point itself. They are called directrices of the first and second kinds respectively.*

It is easy to verify, moreover, that the two directrices are reciprocal polars with respect to the hyperboloid which osculates the two osculating ruled surfaces.

§ 5. *Relations between the four linear complexes.*

If we compute the simultaneous invariants of the four complexes C_1, C_2, C', C'', we find

$$
\begin{aligned}
&(\mathfrak{A}, \mathfrak{B}) = 0, (\mathfrak{A}, \mathfrak{A}') = 0, (\mathfrak{A}, \mathfrak{A}'') = -2^4 a'(a'\theta_v - 2a'_v\theta),\\
&(\mathfrak{B}, \mathfrak{A}') = -2^4 b(b\theta'_u - 2b_u\theta'), (\mathfrak{B}, \mathfrak{A}'') = 0,\\
&(\mathfrak{A}', \mathfrak{A}'') = 0,
\end{aligned}
\tag{71}
$$

so that *each of these four complexes is in involution with two of the others.* Those pairs which are not in involution consist of one complex which osculates

an osculating ruled surface of one kind, and of one complex which osculates an asymptotic curve of the other kind. We notice, moreover, that an asymptotic curve of one kind lies on the osculating ruled surface of the other kind, and we may express this relation by saying that they are united in position. *Those of the four osculating complexes which correspond to an asymptotic curve and to an osculating ruled surface in united position are the ones which are not involutory in general.* If the surface is not a ruled surface, one or both of these pairs of complexes are also in involution only if one or both of the two invariants

$$a'\theta_v - 2a'_v\theta \quad \text{or} \quad b\theta'_u - 2b_u\theta'$$

are equal to zero.

We notice that

$$(\mathfrak{A}, \mathfrak{A}'')^2 - 4\mathfrak{A}\mathfrak{A}'' = 2^{10}a'^4C^2\theta. \tag{72}$$

If this invariant vanishes the complexes C_1 and C'' will have in common a congruence with coincident directrices. However, if a' is equal to zero, or if C vanishes, this congruence is indeterminate. In fact, we have already noticed that if $a' = 0$, the complex C_1 is indeterminate. We now notice further that if $C = 0$ the complexes C_1 and C'' become identical, unless $a'\theta_v - 2a'_v\theta$ is also equal to zero, in which case the complex C_1 is indeterminate. Therefore we may state the following theorem.

Let the osculating ruled surface R_1, of a surface S, have a determinate osculating linear complex C_1, i. e., *let it not be included in a linear congruence. Then not both of the invariants C and $a'\theta_v - 2a'_v\theta$ will vanish. If $C = 0$, this complex C_1 will coincide with that one which osculates the asymptotic curve of S which lies upon R_1. If $C \neq 0$, these two complexes determine a congruence whose directrices coincide if and only if the two branches of the flecnode curve of R_1 coincide.*

There is of course a corresponding theorem with respect to R_2.

Four linear complexes have, in general, two straight lines in common. We proceed to determine the straight lines which belong to the four complexes C_1, C_2, C', C''. We have already found the directrices of the congruence common to the complexes C' and C''. They were the lines d and d' of § 4 (equations (70a) and (70b)). The lines common to the four complexes must intersect d and d'.

The two points

$$p = -a'_u y + 2a'z, \qquad q = -b_v y + 2b_\rho, \tag{73}$$

are on d; the two points

$$y, \qquad r = -a'b_v z - ba'_u\rho + 2a'b\sigma, \tag{74}$$

are on d'. Therefore the coördinates of an arbitrary point on d will be

$$-(\lambda a'_u + \mu b_v)y + 2a'\lambda z + 2b\mu\rho$$

or, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$,

$$x_1 = -(\lambda a'_u + \mu b_v), \qquad x_2 = 2a'\lambda, \qquad x_3 = 2b\mu, \qquad x_4 = 0.$$

Similarly the coördinates of an arbitrary point of d' will be

$$x'_1 = \lambda', \qquad x'_2 = -\mu' a' b_v, \qquad x'_3 = -\mu' b a'_u, \qquad x'_4 = 2\mu' a' b.$$

The Plückerian coördinates of the line joining these points will be

$$\begin{aligned}
\omega_{12} &= -2a'\lambda\lambda' + a'a'_u b_v \lambda\mu' + a' b_v^2 \mu\mu', \\
\omega_{13} &= -2b\mu\lambda' + b a'^2_u \lambda\mu' + b b_v a'_u \mu\mu', \\
\omega_{14} &= \qquad -2a' b a'_u \lambda\mu' - 2a' b b_v \mu\mu', \\
\omega_{23} &= \qquad -2a' b a'_u \lambda\mu' + 2a' b b_v \mu\mu', \\
\omega_{24} &= \qquad 4a'^2 b \lambda\mu', \\
\omega_{34} &= \qquad\qquad 4a' b^2 \mu\mu'.
\end{aligned}$$

This line belongs to the other two complexes C_1 and C_2 if

$$\begin{aligned}
a'^2 b C[(2^6 b_v a'_u - 2^3 a'^2 b^2)\mu\mu' - 2^7 \mu\lambda' + \theta\lambda\mu'] = 0, \\
a' b^2 C'[-2^7 \lambda\lambda' + 2^6 (a'_u b_v - 2^3 a'^2 b^2)\lambda\mu' + \theta'\mu\mu'] = 0.
\end{aligned}$$

Let us leave aside the case in which S is a ruled surface, in which case at least one of the four complexes is indeterminate, and also the case that one or both of the quantities C or C' are equal to zero, since then at least two of the complexes coincide. We shall then have a', b, C and C' all different from zero. The lines which are common to the four complexes must then be those for which

$$\begin{aligned}
&-2^7\lambda\lambda' + 2^6(a'_u b_v - 2^3 a'^2 b^2)\lambda\mu' + \theta'\mu\mu' = 0, \\
&-2^7\mu\lambda' + \theta\lambda\mu' + 2^6(a'_u b_v - 2^3 a'^2 b^2)\mu\mu' = 0,
\end{aligned} \tag{75}$$

whence

$$\lambda^2 : \mu^2 = \theta' : \theta,$$

$$2^7 \frac{\lambda'}{\mu'} = 2^6(a'_u b_v - 8a'^2 b^2) + \theta\frac{\lambda}{\mu}. \tag{76}$$

The four complexes have, therefore, two lines in common unless both θ and θ' are equal to zero in which case they have an infinity of lines in common, which form a plane pencil. In fact equations (76) show that these lines join an arbitrary point of d to the point

$$x'_1 = a'_u b_v - 8a'^2 b^2, \qquad x'_2 = -2a' b_v, \qquad x'_3 = -2b a'_u, \qquad x'_4 = 4a' b$$

of d'. This latter point moreover is the point in which d' intersects the osculating hyperboloid H whose equation is

$$x_1 x_4 - x_2 x_3 + 2a' b x_4^2 = 0.$$

If θ and θ' do not both vanish, we have

$$\begin{aligned} &\lambda : \mu = \pm \sqrt{\theta'} : \sqrt{\theta}, \\ &\lambda' : \mu' = \tfrac{1}{2}(a'_u b_v - 8a'^2 b^2) \pm \frac{1}{2^7}\sqrt{\theta\theta'}; \end{aligned} \tag{77}$$

i. e., the four complexes have two lines in common. They intersect d in two points which form a harmonic group with the points p and q; their intersections with d' are harmonic conjugates with respect to H. We may recapitulate as follows.

Consider a surface S which is not ruled, at a point P for which no two of the four fundamental linear complexes coincide. These four linear complexes have an infinity of lines in commom only in the case that the two branches of the flecnode curve coincide on both of the osculating ruled surfaces. The lines common to the four complexes then form a pencil whose vertex is the point where the directrix of the second kind intersects the osculating hyperboloid, and whose plane is determined by this point and the directrix of the first kind. Moreover the flecnodes of the osculating ruled surfaces are the points where their generators intersect the directrix of the first kind.

In every other case the four complexes have only two lines in common. Let there be marked upon the directrix of the first kind its intersections with the asymptotic tangents of P. The two lines common to the four complexes intersect the directrix of the first kind in two points which are harmonic conjugates of each other with respect to these intersections. They intersect the directrix of the second in two points which are harmonic conjugates of each other with respect to the pair of points in which this directrix intersects the osculating hyperboloid.

§ 6. *Canonical development of the non-homogeneous coördinates of a surface.*

We have the following expressions for the first and second derivatives of y;

$$\begin{aligned} &y_u = z, \qquad y_v = \rho, \\ &y_{uu} = -fy - 2b\rho, \qquad y_{uv} = \sigma, \qquad y_{vv} = -gy - 2a'z, \end{aligned} \tag{78}$$

whence

$$\begin{aligned} y_{uuu} &= f_u y - fz - 2b_u\rho - 2b\sigma, \\ y_{uuv} &= (2bg - f_v)y + 4a'bz - (f + 2b_v)\rho, \\ y_{uvv} &= (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho, \\ y_{vvv} &= -g_v y - 2a'_v z - g\rho - 2a'\sigma. \end{aligned} \tag{79}$$

The expressions for the fourth derivatives of y are capable of several forms which become identical if one takes account of the integrability conditions.

We find

$$
\begin{aligned}
y_{uuuu} &= \alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma, \\
y_{uuuv} &= \beta_1 y + \beta_2 z + \beta_3 \rho + \beta_4 \sigma, \\
y_{uuvv} &= \gamma_1 y + \gamma_2 z + \gamma_3 \rho + \gamma_4 \sigma, \\
y_{uvvv} &= \delta_1 y + \delta_2 z + \delta_3 \rho + \delta_4 \sigma, \\
y_{vvvv} &= \epsilon_1 y + \epsilon_2 z + \epsilon_3 \rho + \epsilon_4 \sigma,
\end{aligned} \tag{80}
$$

where

$$
\begin{aligned}
\alpha_1 &= f^2 - 4b^2 g + 2bf_v - f_{uu}, & \alpha_2 &= -2(f_u + 4a'b^2), \\
\alpha_3 &= 4b(f + b_v) - 2b_{uu}, & \alpha_4 &= -4b_u, \\
\beta_1 &= 2bg_u + 2b_u g - 4a'bf - f_{uv}, & \beta_2 &= 4a'_u b + 4a'b_u + 2bg - f_v, \\
\beta_3 &= -(f_u + 2b_{uv} + 8a'b^2), & \beta_4 &= -(f + 2b_v), \\
\gamma_1 &= 4a'_u f + 2a'f_u + fg - g_{uu}, & \gamma_2 &= 2a'f - 2g_u - 2a'_{uu}, \\
\gamma_3 &= 2bg + 8a'_u b + 4a'b_u, & \gamma_4 &= 4a'b, \\
\delta_1 &= 2a'f_v + 2a'_v f - 4a'bg - g_{uv}, & \delta_2 &= -(g_v + 2a'_{uv} + 8a'^2 b), \\
\delta_3 &= 4b_v a' + 4a'_v b + 2a'f - g_u, & \delta_4 &= -(g + 2a'_u), \\
\epsilon_1 &= g^2 - 4a'^2 f + 2a'g_u - g_{vv}, & \epsilon_2 &= 4a'(g + a'_u) - 2a'_{vv}, \\
\epsilon_3 &= -2(g_v + 4a'^2 b), & \epsilon_4 &= -4a'_v.
\end{aligned} \tag{81}
$$

Assume that the point P of the surface S corresponds to the values (0, 0) of u and v, an assumption which involves no restriction of generality. Let the surface be analytical in the vicinity of this point. Then, for values of u and v sufficiently small, we shall have

$$Y = y + y_u u + y_v v + \tfrac{1}{2}(y_{uu} u^2 + 2y_{uv} uv + y_{vv} v^2) + \cdots,$$

which may be written in the form

$$Y = x_1 y + x_2 z + x_3 \rho + x_4 \sigma$$

by making use of equations (78), (79), (80). The expressions for $x_1, \cdots, x_4$ will be power-series proceeding according to powers of u and v and convergent in a certain domain. Moreover $x_1, \cdots, x_4$ will represent the coördinates of any point P' of the surface S within a certain vicinity of the point P, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$.

We find

$$
\begin{aligned}
x_1 &= 1 - \tfrac{1}{2}fu^2 - \tfrac{1}{2}gv^2 - \tfrac{1}{6}f_u u^3 + \tfrac{1}{2}(2bg - f_v)u^2v + \tfrac{1}{2}(2a'f - g_u)uv^2 \\
&\quad - \tfrac{1}{6}g_v v^3 + \tfrac{1}{24}(\alpha_1 u^4 + 4\beta_1 u^3 v + 6\gamma_1 u^2 v^2 + 4\delta_1 uv^3 + \epsilon_1 v^4) + \cdots, \\
x_2 &= u - a'v^2 - \tfrac{1}{6}fu^3 + 2a'bu^2v - \tfrac{1}{2}(g + 2a'_u)uv^2 - \tfrac{1}{3}a'_v v^3 \\
&\quad + \tfrac{1}{24}(\alpha_2 u^4 + 4\beta_2 u^3 v + 6\gamma_2 u^2 v^2 + 4\delta_2 uv^3 + \epsilon_2 v^4) + \cdots, \\
x_3 &= v - bu^2 - \tfrac{1}{3}b_u u^3 - \tfrac{1}{2}(f + 2b_v)u^2v + 2a'buv^2 - \tfrac{1}{6}gv^3 \\
&\quad + \tfrac{1}{24}(\alpha_3 u^4 + 4\beta_3 u^3 v + 6\gamma_3 u^2 v^2 + 4\delta_3 uv^3 + \epsilon_3 v^4) + \cdot \\
x_4 &= uv - \tfrac{1}{3}bu^3 - \tfrac{1}{3}a'v^3 \\
&\quad + \tfrac{1}{24}(\alpha_4 u^4 + 4\beta_4 u^3 v + 6\gamma_4 u^2 v^2 + 4\delta_4 uv^3 + \epsilon_4 v^4) + \cdots
\end{aligned}
\tag{82}
$$

Introduce non-homogeneous coördinates by putting

$$
\xi = \frac{x_2}{x_1}, \qquad \eta = \frac{x_3}{x_1}, \qquad \zeta = \frac{x_4}{x_1}. \tag{83}
$$

Then

$$
\begin{aligned}
\xi &= u - a'v^2 + \tfrac{1}{3}fu^3 + 2a'bu^2v - a'_u uv^2 - \tfrac{1}{3}a'_v v^3 \\
&\quad + \tfrac{1}{24}[(\alpha_2 + 4f'_u)u^4 + (4\beta_2 - 24bg + 12f_v)u^3v + (6\gamma_2 - 36a'f + 12g_u)u^2v^2 \\
&\qquad + (4\delta_2 + 4g_v)uv^3 + (\epsilon_2 - 12a'g)v^4] + \cdots, \\
\eta &= v - bu^2 - \tfrac{1}{3}b_u u^3 - b_v u^2 v + 2a'buv^2 + \tfrac{1}{3}gv^3 \\
&\quad + \tfrac{1}{24}[(\alpha_3 - 12bf)u^4 + (4\beta_3 + 4f_u)u^3v + (6\gamma_3 - 36bg + 12f_v)u^2v^2 \\
&\qquad + (4\delta_3 - 24a'f + 12g_u)uv^3 + (\epsilon_3 + 4g_v)v^4] + \cdots, \\
\zeta &= uv - \tfrac{1}{3}bu^3 - \tfrac{1}{3}a'v^3 + \tfrac{1}{24}[\alpha_4 u^4 + (4\beta_4 + 12f')u^3v + 6\gamma_4 u^2v^2 \\
&\qquad + (4\delta_4 + 12g)uv^3 + \epsilon_4 v^4] + \cdots,
\end{aligned}
$$

whence

$$
\begin{aligned}
\xi\eta &= uv - bu^3 - a'v^3 - \tfrac{1}{3}b_u u^4 + \tfrac{1}{3}(f - 3b_v)u^3v + 5a'bu^2v^2 \\
&\quad + \tfrac{1}{3}(g - 3a'_u)uv^3 - \tfrac{1}{3}a'_v v^4 + \cdots,
\end{aligned}
$$

and consequently, substituting the values of α_4, β_4, γ_4, δ_4, ϵ_4, from (81),

$$
\begin{aligned}
\zeta - \xi\eta &= \tfrac{2}{3}bu^3 + \tfrac{2}{3}a'v^3 + \tfrac{1}{6}b_u u^4 + \tfrac{2}{3}b_v u^3 v - 4a'bu^2v^2 \\
&\quad + \tfrac{2}{3}a'_u uv^3 + \tfrac{1}{6}a'_v v^4 + \cdots.
\end{aligned}
\tag{84}
$$

We find further, up to terms of the fourth order,

$$
\xi^3 = u^3 - 3a'u^2v^2 + \cdots, \qquad \eta^3 = v^3 - 3bu^2v^2 + \cdots, \qquad \xi^4 = u^4 + \cdots,
$$

$$
\xi^3\eta = u^3v + \cdots, \qquad \xi^2\eta^2 = u^2v^2 + \cdots, \qquad \xi\eta^3 = uv^3 + \cdots, \qquad \eta^4 = v^4 + \cdots,
$$

so that we may write in (84)

$$u^3 = \xi^3 + 3a'\xi^2\eta^2, \qquad v^3 = \eta^3 + 3b\xi^2\eta^2, \qquad u^4 = \xi^4, \text{ etc.}$$

The result is the development, exact up to terms of the fourth order,

$$(85)\quad \zeta = \xi\eta + \tfrac{2}{3}b\xi^3 + \tfrac{2}{3}a'\eta^3 + \tfrac{1}{6}(b_u\xi^4 + 4b_c\xi^3\eta + 4a'_u\xi\eta^3 + a'_c\eta^4) + \cdots.$$

The tetrahedron of reference which gives rise to the development (85) has for two of its edges the asymptotic tangents of the point P, the equations of which are: $\eta = \zeta = 0$ and $\xi = \zeta = 0$. We proceed to make the most general transformation of coördinates which will retain these two lines as edges of the tetrahedron of reference, viz.;

$$(86)\quad x = \frac{\lambda\xi + \lambda'\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta}, \qquad y = \frac{\mu\eta + \mu'\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta}, \qquad z = \frac{\nu\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta},$$

where $\lambda, \lambda', \mu, \mu', \nu, \alpha, \beta, \gamma$ are arbitrary constants, which we shall choose in such a way as to simplify the development of the equation of the surface into series.

We now perform the calculations by substituting in (86) for ζ the series (85). We find

$$\frac{1}{1+\alpha\xi+\beta\eta+\gamma\zeta} = 1 - \alpha\xi - \beta\eta + \alpha^2\xi^2 + (2\alpha\beta - \gamma)\xi\eta + \beta^2\eta^2 - \tfrac{2}{3}b\gamma\xi^3 + 2\alpha\gamma\xi^2\eta + 2\beta\gamma\xi\eta^2 - \tfrac{2}{3}a'\gamma\eta^3 + (\ )_4,$$

where the symbol $(\ )_n$ denotes terms of the nth order. Consequently

$$(87)\quad \begin{aligned} x = {} & \lambda\xi - \lambda\alpha\xi^2 + (\lambda' - \lambda\beta)\xi\eta + (\lambda\alpha^2 + \tfrac{2}{3}\lambda' b)\xi^3 \\ & + [\lambda(2\alpha\beta - \gamma) - \lambda'\alpha]\xi^2\eta + \beta(\lambda\beta - \lambda')\xi\eta^2 + \tfrac{2}{3}\lambda'a'\eta^3 + (\ )_4, \end{aligned}$$

whence

$$(88)\quad \begin{aligned} x^2 &= \lambda^2\xi^2 - 2\lambda^2\alpha\xi^3 + 2\lambda(\lambda' - \lambda\beta)\xi^2\eta + (\ )_4, \\ x^3 &= \lambda^3\xi^3 - 3\lambda^3\alpha\xi^4 + 3\lambda^2(\lambda' - \lambda\beta)\xi^3\eta + (\ )_5, \\ x^4 &= \lambda^4\xi^4 + (\ )_5. \end{aligned}$$

Similarly

$$\begin{aligned} y = {} & \mu\eta + (\mu' - \mu\alpha)\xi\eta - \mu\beta\eta^2 + \tfrac{2}{3}\mu' b\xi^3 + \alpha(\mu\alpha - \mu')\xi^2\eta \\ & + [\mu(2\alpha\beta - \gamma) - \mu'\beta]\xi\eta^2 + (\mu\beta^2 + \tfrac{2}{3}\mu'a')\eta^3 + (\ )_4, \end{aligned}$$

$$(89)\quad \begin{aligned} y^2 &= \mu^2\eta^2 + 2\mu(\mu' - \mu\alpha)\xi\eta^2 - 2\mu^2\beta\eta^3 + (\ )_4, \\ y^3 &= \mu^3\eta^3 + 3\mu^2(\mu' - \mu\alpha)\xi\eta^3 - 3\mu^3\beta\eta^4 + (\ )_5, \\ y^4 &= \mu^4\eta^4 + (\ )_5, \end{aligned}$$

and

$$(90)\quad x^3y = \mu\lambda^3\xi^3\eta + (\ )_5, \quad x^2y^2 = \lambda^2\mu^2\xi^2\eta^2 + (\ )_5, \quad xy^3 = \lambda\mu^3\xi\eta^3 + (\ )_5.$$

We have further

$$(91)\qquad \frac{1}{\nu}z=\xi\eta+\tfrac{2}{3}b\xi^3-\alpha\xi^2\eta-\beta\xi\eta^2+\tfrac{2}{3}a'\eta^3+(\tfrac{1}{6}b_u-\tfrac{2}{3}b\alpha)\xi^4+(\alpha^2-\tfrac{2}{3}b\beta+\tfrac{2}{3}b_v)\xi^3\eta$$
$$+(2\alpha\beta-\gamma)\xi^2\eta^2+(\beta^2-\tfrac{2}{3}a'\alpha+\tfrac{2}{3}a'_u)\xi\eta^3+(\tfrac{1}{6}a'_v-\tfrac{2}{3}a'\beta)\eta^4+(\ )_5,$$

$$\frac{xy}{\lambda\mu}=\xi\eta+\left(\frac{\mu'}{\mu}-2\alpha\right)\xi^2\eta+\left(\frac{\lambda'}{\lambda}-2\beta\right)\xi\eta^2+\tfrac{2}{3}\frac{\mu'}{\mu}b\xi^4$$
$$+\left(3\alpha^2+\tfrac{2}{3}\frac{\lambda'}{\lambda}b-2\frac{\mu'}{\mu}\alpha\right)\xi^3\eta$$
$$+\left(6\alpha\beta-2\gamma-2\frac{\lambda'}{\lambda}\alpha-2\frac{\mu'}{\mu}\beta+\frac{\lambda'}{\lambda}\frac{\mu'}{\mu}\right)\xi^2\eta^2$$
$$+\left(3\beta^2+\tfrac{2}{3}\frac{\mu'}{\mu}a'-2\frac{\lambda'}{\lambda}\beta\right)\xi\eta^3+\tfrac{2}{3}\frac{\lambda'}{\lambda}a'\eta^4+(\ )_5;$$

and we find from (88) and (89)

$$\frac{x^3}{\lambda^3}=\xi^3-3\alpha\xi^4+3\left(\frac{\lambda'}{\lambda}-\beta\right)\xi^3\eta+(\ )_5,$$

$$\frac{x^2y}{\lambda^2\mu}=\xi^2\eta+\left(\frac{\mu'}{\mu}-3\alpha\right)\xi^3\eta+\left(2\frac{\lambda'}{\lambda}-3\beta\right)\xi^2\eta^2+(\ )_5,$$

$$\frac{xy^2}{\lambda\mu^2}=\xi\eta^2+\left(2\frac{\mu'}{\mu}-3\alpha\right)\xi^2\eta^2+\left(\frac{\lambda'}{\lambda}-3\beta\right)\xi\eta^3+(\ )_5,$$

$$\frac{y^3}{\mu^3}=\eta^3+3\left(\frac{\mu'}{\mu}-\alpha\right)\xi\eta^3-3\beta\eta^4+(\ )_5.$$

The development (85) contains of the third degree only the terms ξ^3 and η^3. If we now compute the development for z in terms of x and y we find that there will be introduced, in general, terms of the form x^2y and xy^2. In fact we find that

$$\frac{z}{\nu}-\frac{xy}{\lambda\mu}-\tfrac{2}{3}b\frac{x^3}{\lambda^3}-\left(\alpha-\frac{\mu'}{\mu}\right)\frac{x^2y}{\lambda^2\mu}-\left(\beta-\frac{\lambda'}{\lambda}\right)\frac{xy^2}{\lambda\mu^2}-\tfrac{2}{3}a'\frac{y^3}{\mu^3}$$

involves only terms of the fourth order. The form of the development (85) will be preserved if and only if

$$(92)\qquad \frac{\mu'}{\mu}=\alpha,\qquad \frac{\lambda'}{\lambda}=\beta.$$

Assuming these relations to be satisfied we find

$$\frac{z}{\nu}-\frac{xy}{\lambda\mu}-\tfrac{2}{3}b\frac{x^3}{\lambda^3}-\tfrac{2}{3}a'\frac{y^3}{\mu^3}=(\tfrac{1}{6}b_u+\tfrac{2}{3}b\alpha)\frac{x^4}{\lambda^4}+(\tfrac{2}{3}b_v-\tfrac{4}{3}b\beta)\frac{x^3y}{\lambda^3\mu}$$
$$+(\gamma-\alpha\beta)\frac{x^2y^2}{\lambda^2\mu^2}+(\tfrac{2}{3}a'_u-\tfrac{4}{3}a'\alpha)\frac{xy^3}{\lambda\mu^3}+(\tfrac{1}{6}a'_v+\tfrac{2}{3}a'\beta)\frac{y^4}{\mu^4}.$$

Let neither a' nor b be zero at the point considered. In fact a' or b can vanish only at special points of the surface unless it is a ruled surface. We may then determine α, β and γ so as to make the coefficients of x^3y, x^2y^2, and xy^3 disappear. This gives

$$\alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}, \qquad \gamma = \alpha\beta = \frac{a'_u b_v}{4a'b}. \tag{93}$$

The development now becomes

$$z = \frac{\nu}{\lambda\mu}xy + \tfrac{2}{3}b\frac{\nu}{\lambda^3}x^3 + \tfrac{2}{3}a'\frac{\nu}{\mu^3}y^3 + \tfrac{1}{6}\left(b_u + 2b\frac{a'_u}{a'}\right)\frac{\nu}{\lambda^4}x^4 + \tfrac{1}{6}\left(a'_v + 2a'\frac{b_v}{b}\right)\frac{\nu}{\mu^4}y^4 + \cdots.$$

Let λ, μ, ν be chosen as non-vanishing quantities so that

$$\lambda^3 = 4b\nu, \qquad \mu^3 = 4a'\nu, \qquad \nu = \lambda\mu. \tag{94}$$

Then

$$\lambda^2 = 4b\mu, \qquad \mu^2 = 4a'\lambda.$$

whence

$$\lambda = 4\sqrt[3]{A}, \qquad \mu = 4\sqrt[3]{B}, \qquad \nu = 16a'b, \tag{95}$$

where

$$A = a'b^2, \qquad B = a'^2b, \tag{95a}$$

and where the symbol $\sqrt[3]{A}$ may be taken to have any one of its three values, the corresponding value of $\sqrt[3]{B}$ being then determined from the relation

$$4b\mu = \lambda^2.$$

We thus find the canonical development

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots, \tag{96}$$

where I and J,

$$I = \frac{B_u}{4B\sqrt[3]{A}}, \qquad J = \frac{A_v}{4A\sqrt[3]{B}}, \tag{96a}$$

are absolute invariants. All further coefficients will be absolute invariants.

This development was given for the first time in 1893 by TRESSE in his Paris thesis. His proof, however, is entirely different from the above, and the geometrical significance of this development, which we shall now proceed to study, is not found in TRESSE's thesis nor, so far as I am aware, elsewhere.

The transformation which connects the tedrahedron $P_yP_zP_\rho P_\sigma$ with the tetrahedron, for which the development assumes the canonical form, is

$$x = \frac{\lambda(\xi+\beta\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad y = \frac{\mu(\eta+\alpha\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad z = \frac{\nu\zeta}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \tag{97}$$

where

$$(97a)\qquad \alpha=\frac{a'_u}{2a'},\qquad \beta=\frac{b_v}{2b},\qquad \lambda=4\sqrt{A},\qquad \mu=4\sqrt[3]{B},$$

$$\nu=\lambda\mu=16a'b,\qquad A=a'b^2,\qquad B=a'^2b.$$

If we write

$$(98)\qquad x=\frac{X}{\omega},\qquad y=\frac{Y}{\omega},\qquad z=\frac{Z}{\omega},$$

so as to introduce homogeneous coördinates, and make use of (83), if moreover we denote by κ a factor of proportionality, we shall have;

$$(99)\qquad \kappa X=\lambda(x_2+\beta x_4),\qquad \kappa Y=\mu(x_3+\alpha x_4),\qquad \kappa Z=\lambda\mu x_4,$$

$$\kappa\omega=x_1+\alpha x_2+\beta x_3+\alpha\beta x_4,$$

whence

$$(100)\qquad \frac{\lambda\mu}{\kappa}x_1=-\alpha\mu X-\lambda\beta Y+\alpha\beta Z+\lambda\mu\omega,$$

$$\frac{\lambda\mu}{\kappa}x_2=\mu X-\beta Z,\qquad \frac{\lambda\mu}{\kappa}x_3=\lambda Y-\alpha Z,\qquad \frac{\lambda\mu}{\kappa}x_4=Z.$$

The plane $Z=0$ is the plane tangent to the surface at the given point; the lines $X=Z=0$ and $Y=Z=0$ are the two asymptotic tangents at this point. The line $\omega=Z=0$ according to $(70a)$ is the directrix of the first kind. The line $X=Y=0$ is the directrix of the second kind. We have found, therefore, the geometrical significance of three of the faces, of three of the vertices, and of four of the edges of our canonical tetrahedron. We shall be able to describe it completely as a result of the investigation contained in the following paragraph, which is based upon the consideration of a certain class of surfaces of the third order.

§ 7. *The canonical cubic of a point of the surface.*

Let the surface S be referred to the canonical tetrahedron which belongs to one of its points P, and consider the cubic surface

$$(101)\qquad z=xy+\tfrac{1}{6}(x^3+y^3),$$

which shall be called the canonical cubic of the point. Its equation in the homogeneous coördinates (98) is

$$(102)\qquad F=6Z\omega^2-6XY\omega-X^3-Y^3=0.$$

We find

$$F_X=-6Y\omega-3X^2,\quad F_Y=-6X\omega-3Y^2,\quad F_Z=6\omega^2,\quad F_\omega=12Z\omega-6XY,$$

where

$$F_X = \frac{\partial F}{\partial X},$$

etc. Therefore, the only singular point of this surface is

$$X = Y = \omega = 0.$$

Moreover, the examination of the second derivatives shows that this point is a *unode;* i. e., the quadric cone which, in general, replaces the tangent plane at a double point, degenerates into a pair of coincident planes. These planes, moreover, in this case coincide with the plane $\omega = 0$, which is, therefore, the *uniplane* of the cubic surface.

What is the exact and characteristic relation of the cubic surface (102) to the surface S? We verify without difficulty that the most general cubic surface which has the point $X = Y = \omega = 0$, $Z = 1$, as a unode and the plane $\omega = 0$ as uniplane is

$$F \equiv Z\omega^2 + \phi(X, Y, \omega) = 0, \tag{103}$$

where $\phi(X, Y, \omega)$ is a general ternary cubic in X, Y, ω, so that the equation contains ten arbitrary constants. Let the uniplane, instead of being $\omega = 0$, be the arbitrary plane

$$\omega + pX + qY + rZ = 0,$$

and let the unode be an arbitrary point

$$X = -\lambda, \qquad Y = -\mu, \qquad Z = 1, \qquad \omega = p\lambda + q\mu - r,$$

of this plane; the equation of the most general unodal cubic surface will then be obtained by putting

$$\omega + pX + qY + rZ, \qquad X + \lambda Z, \qquad Y + \mu Z, \qquad Z,$$

in place of

$$\omega, \qquad X, \qquad Y, \qquad Z,$$

respectively in (103). We thus find

$$F \equiv Z(\omega + pX + qY + rZ)^2 + \phi(X + \lambda Z,\ Y + \mu Z,\ \omega + pX + qY + rZ) = 0, \tag{104}$$

an equation involving fifteen arbitrary constants as that of the most general unodal cubic or, in non-homogeneous coördinates,

$$z(1 + px + qy + rz)^2 + \phi(x + \lambda z,\ y + \mu z,\ 1 + px + qy + rz) = 0. \tag{105}$$

We shall assume in the first place that this cubic surface has contact of the third order with S at the point P. Then, in particular, its asymptotic tangents at P will coincide with those of S. The plane $z = 0$ must therefore intersect the cubic in a cubic curve which has P as a double point, the asymptotic tangents of P being the tangents of the double point.

The plane $Z = 0$ intersects the surface (103) in the cubic curve

$$\phi(X, Y, \omega) = 0,$$

or in the notation of SALMON*

$$aX^3 + bY^3 + c\omega^3 + 3a_2X^2Y + 3a_3X^2\omega + 3b_1Y^2X + 3b_3Y^2\omega + 3c_1\omega^2X + 3c_2\omega^2Y + 6mXY\omega = 0.$$

In order that the point $X = Y = 0$ may be a double point with the lines $XY = 0$ as tangents, we must have

$$c = c_1 = c_2 = a_3 = b_3 = 0,$$

so that, with a slight change of notation, we may write

$$\phi(X, Y, \omega) = aX^3 + bY^3 + 3a'X^2Y + 3b'XY^2 + 6mXY\omega.$$

Introducing non-homogeneous coördinates, we find as the equation of the most general unodal cubic which touches the surface S at the point P and whose asymptotic tangents at that point coincide with those of S, the following:

$$(106a) \qquad z(1 + px + qy + rz)^2 + \phi(x + \lambda z, y + \mu z, 1 + px + qy + rz) = 0,$$

where

$$(106b) \qquad \begin{aligned}\phi = {} & a(x+\lambda z)^3 + b(y+\mu z)^3 + 3a'(x+\lambda z)^2(y+\mu z) + 3b'(x+\lambda z)(y+\mu z)^2 \\ & + 6m(x+\lambda z)(y+\mu z)(1+px+qy+rz).\end{aligned}$$

If this surface has contact of the third order with the surface S at the point P, the substitution of the development

$$(96) \qquad z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots$$

into (106) must satisfy these equations up to terms of the third order inclusive. Since we shall need them for a later investigation, we shall compute the terms of the fourth order which result from the substitution of (96) into (106), as well as those of the third.

We find

$$(107) \qquad \begin{aligned} & 1 + px + qy + rz = 1 + px + qy + rxy + \frac{r}{6}(x^3+y^3) + \frac{r}{24}(Ix^4+Jy^4) + \cdots, \\ & z(1+px+qy+rz)^2 = xy + \tfrac{1}{6}(x^3 + 12px^2y + 12qxy^2 + y^3) + (\tfrac{1}{24}I + \tfrac{1}{3}p)x^4 \\ & \qquad + (\tfrac{1}{3}q + p^2)x^3y + 2(pq + r)x^2y^2 + (\tfrac{1}{3}p + q^2)xy^3 + (\tfrac{1}{24}J + \tfrac{1}{3}q)y^4 + \cdots, \\ & (x + \lambda z)^2(y + \mu z) = x^2y + 2\lambda x^2y^2 + \mu x^3y + \cdots, \\ & (x + \lambda z)(y + \mu z)^2 = xy^2 + 2\mu x^2y^2 + \lambda xy^3 + \cdots, \\ & (x+\lambda z)(y+\mu z)(1+px+qy+rz) = xy + (p+\mu)x^2y + (q+\lambda)xy^2 + \tfrac{1}{6}\mu x^4 \\ & \qquad + (\tfrac{1}{6}\lambda + p\mu)x^3y + (\lambda\mu + p\lambda + q\mu + r)x^2y^2 + (\tfrac{1}{6}\mu + q\lambda)xy^3 + \tfrac{1}{6}\lambda y^4 + \cdots. \end{aligned}$$

* Higher plane curves, 3d edition, p. 189.

This gives, on substitution in the left member of (106),

$$(1+6m)xy+(a+\tfrac{1}{6})x^3+[2p+3a'+6m(\mu+p)]x^2y$$
$$+[2q+3b'+6m(\lambda+q)]xy^2+(b+\tfrac{1}{6})y^3,$$
$$(108)\quad +(\tfrac{1}{24}I+\tfrac{1}{3}p+m\mu)x^4+(p^2+\tfrac{1}{3}q+3a\lambda+3a'\mu+m\lambda+6mp\mu)x^3y$$
$$+[2(pq+r)+6b'\mu+6a'\lambda+6m(\lambda\mu+p\lambda+q\mu+r)]x^2y^2,$$
$$+(q^2+\tfrac{1}{3}p+3b\mu+3b'\lambda+m\mu+6mq\lambda)xy^3+(\tfrac{1}{24}J+\tfrac{1}{3}q+m\lambda)y^4+\cdots.$$

If the surface (106) has contact of the third order with S at P, the terms of the third order in (108) must vanish, i. e.,

$$(109)\quad a=-\tfrac{1}{6},\quad b=-\tfrac{1}{6},\quad m=-\tfrac{1}{6},\quad p+3a'-\mu=0,\quad q+3b'-\lambda=0.$$

The uniplane of the unodal cubic surface will pass through the directrix d of the first kind if and only if

$$p=q=0,$$

since the equations of d are $Z=\omega=0$ in homogeneous form. Assuming this to be the case, the unode will be the intersection of the three planes

$$\omega+rZ=0,\qquad X+\lambda Z=0,\qquad Y+\mu Z=0.$$

This can be a point of the directrix of the second kind d' if and only if $\lambda=\mu=0$, since the equations of d' are

$$X=Y=0.$$

If we combine these conditions with (129), we notice that to every ordinary point P of a surface S there belongs a family of ∞^1 unodal cubic surfaces, each of which has contact of the third order with S at P, whose uniplane contains the directrix of the first kind and whose unode is on the directrix of the second kind of the point P. The equation of the most general unodal cubic of this kind is

$$(110)\qquad z(1+rz)^2-\tfrac{1}{6}x^3-\tfrac{1}{6}y^3-xy(1+rz)=0.$$

Consider a surface U of this family; it has contact of the third order with S at P, i. e., *every* plane which contains P will intersect the surfaces U and S in plane curves which have contact of the third order with each other at P. We can determine four tangents through P which may be called tangents of fourth order contact with respect to U. For, every plane which passes through such a tangent will intersect S and U in two plane curves which have at least fourth order contact at P. In order to find these tangents we introduce the values of the coefficients, a, b, etc., for the surface U into (108) and equate to zero the terms of the fourth order which do not vanish identically. We thus find that the equation

$$Ix^4 + 24rx^2y^2 + Jy^4 = 0$$

determines the four tangents of fourth order contact between U and S.

The condition that the quartic

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = 0$$

may represent a harmonic pencil is

$$j = ace + 2bcd - ad^2 - b^2e - c^3 = 0.$$

The four tangents of fourth order contact between U and S will therefore form a harmonic group if and only if

$$4IJr - 64r^3 = 0,$$

i. e., if

$$r = 0 \qquad \text{or} \qquad r = \pm \tfrac{1}{4}\sqrt{IJ}.$$

Only in the case $r = 0$ will it be possible moreover to arrange these tangents into pairs which divide each other harmonically and which are at the same time harmonic conjugates with respect to the asymptotic tangents of the point P. But for $r = 0$, equation (110) becomes identical with (101), the canonical cubic.

The canonical cubic of a point P is, therefore, completely characterized by the following properties.

1. *It has a unode situated upon the directrix of the second kind of the point P.*

2. *Its uniplane passes through the directrix of the first kind of the point P.*

3. *It has contact of the third order with the given surface S at the point P.*

4. *The four tangents of fourth order contact, between it and the surface S at the point P, form a harmonic pencil.*

5. *If these four tangents be arranged in pairs which divide each other harmonically, the members of each pair are harmonic conjugates also with respect to the asymptotic tangents of P.*

The plane $z = 0$ intersects the canonical cubic in the plane cubic curve

$$x^3 + y^3 + 6xy = 0,$$

or in homogeneous coördinates

$$f \equiv X^3 + Y^3 + 6XY\omega = 0. \tag{111}$$

This plane cubic curve has, as we already know, the point $X = Y = 0$ as double point, the two tangents to the curve at this point being the asymptotic tangents $X = 0$ and $Y = 0$ of the surface S. The Hessian of f, omitting a numerical factor, is

$$H \equiv X^3 + Y^3 - 2XY\omega = 0. \tag{112}$$

The two curves intersect in the double point $X = Y = 0$, which counts for six intersections. The remaining three points of intersection, viz.

$$(113)\quad X=-1,\ Y=1,\ \omega=0;\quad X=-\epsilon,\ Y=1,\ \omega=0;\quad X=-\epsilon^2,\ Y=1,\ \omega=0,$$

where ϵ is an imaginary third root of unity, are all on the line $\omega = 0$. They are the points of inflection of $f = 0$. Therefore we have a further property of the canonical cubic.

The tangent plane intersects the canonical cubic surface in a plane cubic curve which has the point of contact as double point and the asymptotic tangents of this point as tangents. Its three points of inflection are situated upon the directrix of the first kind.

The plane $\omega = 0$, and therefore the canonical tetrahedron, has now been determined. In order to complete the geometrical determination of the system of coördinates which leads to the canonical development, it remains to determine the arbitrary numerical factors. This may be done as follows.

Consider the point of inflection:

$$X=1,\qquad Y=1,\qquad \omega=0,$$

of the plane cubic $f = 0$. The tangent to the curve at that point will be

$$Z=0,\qquad X+Y-2\omega=0.$$

The hyperboloid H, which osculates R_1 and R_2, has the equation

$$8(Z\omega - XY) + Z^2 = 0,$$

when referred to the canonical system of coördinates. The line $X = Y = 0$ intersects this quadric for $Z = 0$ and for $Z + 8\omega = 0$, i. e., in the point of contact and in the further point

$$X=0,\qquad Y=0,\qquad Z=-8,\qquad \omega=1.$$

The plane

$$4(X+Y-2\omega)-Z=0$$

contains this point and passes through the inflectional tangent of the plane cubic which was considered above.

We may recapitulate our results as follows:

Two of the edges of the canonical tetrahedron of a regular point P of a non-ruled surface are the asymptotic tangents of that point. Two other edges are the directrices of the first and second kind which belong to the point P. The face of the tetrahedron opposite to P is the uniplane of its canonical cubic surface. This completes the determination of the tetrahedron. The system of coördinates which gives rise to the canonical development refers to this tetrahedron. The numerical factors which still remain arbitrary in a

projective system of coördinates, after the tetrahedron of reference has been chosen, are determined in such a way that the equation

$$4(x+y-2)-z=0 \tag{114}$$

represents a plane, passing through the second point of intersection of the osculating hyperboloid with the directrix of the second kind, as well as through one of the inflectional tangents of the plane curve in which the tangent plane intersects the canonical cubic surface. Since this plane cubic curve has three inflectional tangents, the canonical development may be made in three ways. It is for this reason that there enters into its expression the cube root of a rational invariant.

The quartic

$$Ix^4+Jy^4=0 \tag{115}$$

determines the tangents of fourth-order contact between the surface S and its canonical cubic. The expressions for I and J (96a) show, therefore, that if A is a function of u alone, or if B is a function of v alone, these four tangents coincide with each other and with one of the two asymptotic tangents of the point considered.

§ 8. *Osculating unodal cubic surfaces.*

Since the most general unodal cubic surface depends upon fifteen arbitrary constants, and since the requirement that one surface shall have contact of the fourth order with another is equivalent to fifteen conditions, there will be a finite number of unodal cubics which have contact of the fourth order with the surface S at a given point P. We shall speak of them as its *osculating unodal cubic surfaces.*

If equations (106) represent such an osculating unodal cubic, the coefficients in the expression (108), as far as computed, must all be equal to zero. This gives

$$\begin{gathered}
a=b=m=-\tfrac{1}{6}, \\
p-\mu+3a'=0, \qquad 8p-4\mu+I=0, \\
q-\lambda+3b'=0, \qquad 8q-4\lambda+J=0, \\
p^2+\tfrac{1}{3}q-\tfrac{2}{3}\lambda+3a'\mu-p\mu=0, \\
q^2+\tfrac{1}{3}p-\tfrac{2}{3}\mu+3b'\lambda-q\lambda=0, \\
2pq+r+6b'\mu+6a'\lambda-\lambda\mu-p\lambda-q\mu=0,
\end{gathered} \tag{116}$$

whence

$$\begin{gathered}
\lambda=2q+\tfrac{1}{4}J, \qquad \mu=2p+\tfrac{1}{4}I, \qquad 3a'=p+\tfrac{1}{4}I, \\
3b'=q+\tfrac{1}{4}J, \qquad r=-2pq-\tfrac{3}{4}(Iq+Jp+\tfrac{1}{4}IJ),
\end{gathered} \tag{117}$$

where

$$p^2 - q - \tfrac{1}{6}J + \tfrac{1}{2}Ip + \tfrac{1}{16}I^2 = 0,$$
$$q^2 - p - \tfrac{1}{6}I + \tfrac{1}{2}Jq + \tfrac{1}{16}J^2 = 0,$$

or

$$(p + \tfrac{1}{4}I)^2 - q - \tfrac{1}{6}J = 0, \tag{118}$$
$$(q + \tfrac{1}{4}J)^2 - p - \tfrac{1}{6}I = 0.$$

Equations (118) will determine, in general, four pairs of values p, q to which will correspond therefore *four osculating unodal cubics.*

Put

$$p + \tfrac{1}{4}I = p', \qquad q + \tfrac{1}{4}J = q', \tag{119}$$

so that (118) becomes

$$p'^2 - q' + \tfrac{1}{12}J = 0, \qquad q'^2 - p' + \tfrac{1}{12}I = 0, \tag{120}$$

whence the biquadratic

$$p'^4 + \tfrac{1}{6}Jp'^2 - p' + \tfrac{1}{12}I + \tfrac{1}{144}J^2 = 0 \tag{121}$$

for p'. Let $p'_1, \cdots p'_4$ be its four roots. The uniplanes of the four osculating unodal cubics will be

$$1 + p_k x + q_k y + r_k z = 0 \qquad (k = 1, 2, 3, 4) \tag{122}$$

where

$$p_k = p'_k - \tfrac{1}{4}I, \qquad q_k = q'_k - \tfrac{1}{4}J = p'^2_k - \tfrac{1}{6}J,$$
$$r_k = -2p'^3_k - \tfrac{1}{4}Ip'^2_k - \tfrac{5}{12}Jp'_k + \tfrac{1}{24}IJ, \qquad (k = 1, 2, 3, 4). \tag{123}$$

The determinant of the four planes (122), writing only one term of each column, is

$$\Delta = |1, p_k, q_k, r_k| = |1, p'_k - \tfrac{1}{4}I, p'^2_k - \tfrac{1}{6}J,$$
$$-2p'^3_k - \tfrac{1}{4}Ip'^2_k - \tfrac{5}{12}Jp'_k + \tfrac{1}{24}IJ|, \tag{124}$$
$$= (-2)|1, p'_k, p'^2_k, p'^3_k|.$$

The determinant in the right member is the square root of the discriminant of the biquadratic (121). It is different from zero unless (121) has a pair of coincident roots. Under the same condition Δ will, therefore, be different from zero.

Hence, *unless two of the four osculating unodal cubic surfaces coincide, their four uniplanes form a non-degenerate tetrahedron.*

Moreover, the vertices of this tetrahedron are not in general identical with the four unodes, which form a second non-degenerate tetrahedron.

§ 9. *Geometric interpretation of the semi-covariants.*

We are now in a position to see just what is the geometrical significance of the fundamental semi-covariants z, ρ and σ.

The equation of the osculating hyperboloid referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ was found to be

$$x_1 x_4 - x_2 x_3 + 2a'bx_4^2 = 0. \tag{11}$$

Therefore, the quadrics of the one-parameter family:

$$x_1 x_4 - x_2 x_3 + \lambda x_4^2 = 0 \tag{125}$$

all have contact of the second order with the surface S at the point P and are tangent to each other along the asymptotic tangents of P. There is just one surface of this family which is tangent to the uniplane of the canonical cubic, the equation of which is

$$x_1 + \alpha x_2 + \beta x_3 + \alpha\beta x_4 = 0, \qquad \alpha = \frac{a'}{2a'}, \qquad \beta = \frac{b}{2b}.$$

In fact, if $(x_1', \cdots x_4')$ is a point of (125) its tangent plane is

$$x_4' x_1 - x_3' x_2 - x_2' x_3 + (x_1' + 2\lambda x_4') x_4 = 0.$$

If this is identical with the above plane, we must have

$$-\frac{x_3'}{x_4'} = \alpha, \qquad -\frac{x_2'}{x_4'} = \beta, \qquad \frac{x_1'}{x_4'} + 2\lambda = \alpha\beta,$$

whence

$$x_1' x_4' - x_2' x_3' + 2\lambda x_4'^2 = 0.$$

But, since the point $(x_1', \cdots x_4')$ is on the quadric (125), we have also

$$x_1' x_4' - x_2' x_3' + \lambda x_4'^2 = 0,$$

so that λ must be equal to zero.

Therefore, *there exists a quadric which is uniquely determined by the following properties:*

1. *It has contact of the second order with the surface S at the point P.*

2. *It touches the osculating hyperboloid H, at all of the points of the asymptotic tangents of P.*

3. *It is tangent to the uniplane of the canonical cubic of the point P.*

The equation of this quadric is

$$x_1 x_4 - x_2 x_3 = 0, \tag{126}$$

or in canonical coördinates

$$\zeta - \xi\eta = 0.$$

It is, therefore, the quadric obtained by omitting all but the quadratic terms in the canonical development of S in the vicinity of the point P. We shall, therefore, speak of it as the *canonical quadric* of P.

The points P_z and P_ρ are obviously situated on the asymptotic tangents of P_y, one on each of them. We now notice that the lines P_zP_σ, $P_\rho P_\sigma$ are the generators of the canonical quadric which pass through P_z and P_ρ respectively, so that P_σ is defined as their point of intersection.

Consider the ruled surface R_1 which is generated by P_yP_z if v alone is variable. The point P_z describes a curve C_z on this ruled surface, whose tangent is obtained by joining P_z to P_σ, since $z_v = \sigma$. For every value of u we obtain such a ruled surface, and upon each of them there is a one-parameter family of such curves C_z. Altogether we obtain in this way a double infinity of curves associated with S, situated upon the osculating ruled surfaces of the first kind. We shall call them the *characteristics of* S of the first kind. The characteristics of the second kind are defined in a similar manner.

We have shown in the first memoir * that the transformation

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

changes y, z, ρ, σ, into

$$\bar{y} = y, \qquad \bar{z} = \frac{1}{\alpha_u}(z + \tfrac{1}{2}\eta y), \qquad \bar{\rho} = \frac{1}{\beta_v}(\rho + \tfrac{1}{2}\zeta y),$$

(127)

$$\bar{\sigma} = \frac{1}{\alpha_u \beta_v}(\sigma + \tfrac{1}{2}\eta\rho + \tfrac{1}{2}\zeta z + \tfrac{1}{4}\eta\zeta y),$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha}, \qquad \zeta = \frac{\beta_{vv}}{\beta}, \tag{128}$$

$\alpha(u)$ and $\beta(v)$ being arbitrary functions of their respective arguments. It will be noticed that, by means of these equations, any characteristic may be transformed into any other of the same kind.

Moreover, if u and v are allowed to vary simultaneously, P_z and P_ρ will describe two surfaces S_z and S_ρ. We shall call S_z the derivative of S_y with respect to u; S_ρ the derivative of S_y with respect to v. The surface S_z is obviously the locus of a single infinity of characteristics of the first kind. The function $\alpha(u)$ may be chosen in such a way that the transformed surface $S_{\bar{z}}$ shall coincide with an arbitrary surface made up of characteristics of the first kind. In fact, the intersection of this surface with the developable of the asymptotic curve $v = \text{const.}$, which passes through P_y, is an arbitrary curve on this developable, since η is an arbitrary function of u.

Thus a system of form (1) not only defines its integral surfaces, but two families of ∞^2 curves, its characteristics. Two other surfaces, each being the locus of a single infinity of characteristics, are associated with every integral surface of (1) as its derivatives with respect to u and v respectively. The independent variables of the system may be chosen in such a way that these two

* These Transactions, vol. 8 (1907), p. 256, formulæ (68).

derivatives become arbitrarily assigned surfaces which are loci of characteristics of the first and of the second kind respectively. If a surface S is given, together with its derivatives with respect to u and v, the independent variables are thereby determined except for a linear transformation. The derivatives serve therefore as geometrical images for the independent variables of the system.

Equations (127) show that the asymptotic curves of S_y are contained as limiting case among the characteristics of the surface. In the case of a ruled surface one of the above families of ∞^2 characteristics consists of straight lines; it gives rise to the flecnode congruence of the ruled surface.* The other degenerates into the one-parameter family of curved asymptotic lines.

The surface S_σ, the locus of P_σ, as u and v assume all of their values, may be called the second derivative of S_y with respect to u and v. We have already noticed that the point P_σ is the intersection of the generators of the canonical quadric which pass through P_z and P_ρ respectively. Clearly, P_σ by itself suffices to determine the two points P_z and P_ρ, so that S_σ *is a sufficient image of the two independent variables.* In other words, let there be associated with every point of a surface S a point of its canonical quadric. If the surface which is the locus of these points be regarded as the second derivative of S with respect to two independent variables u and v, the latter are completely determined thereby, except for a linear transformation.

§ 10. *The directrix congruences and the directrix curves of a surface.*

We have seen in § 4 that the osculating linear complexes of the two asymptotic curves, which meet in a point P of the surface S, have a congruence in common whose directrices we denoted by d and d'. One of them, the directrix of the first kind, denoted by d, is situated in the plane tangent to S at P. The other, that of the second kind, denoted by d', passes through the point P.

We shall first consider the directrix of the second kind d'. Referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ its equations are

$$2bx_2 + b_v x_4 = 0, \qquad 2a'x_3 + a'_u x_4 = 0. \tag{70b}$$

It passes through P_y and the point P_τ where

$$\tau = -a'b_v z - ba'_u \rho + 2a'b\sigma. \tag{129}$$

There exists such a line d' for every point P of the surface S, for which the osculating linear complexes of the two asymptotic curves are determinate and distinct. The totality of these lines forms a congruence, one line of which passes through every (general) point of the surface. We shall call it the *directrix congruence of the second kind.*

* Proj. Diff. Geom., p. 175.

If the point P describes an arbitrary curve on S, its directrix of the second kind d' generates a ruled surface of this congruence. We ask ourselves this question: Can we determine a curve on S such that the ruled surface made up of the directrices of the second kind of its points, shall be a developable? According to the general theory of congruences we know that there will be, in general, two one-parameter families of curves of this kind, because the lines of a congruence can be assembled into a single infinity of developables in two ways. The curves on S obtained in this way shall be called the *directrix curves of the second kind*. Through every point P which is not special, i. e., for which the lines d and d' are not indeterminate, there will pass two such curves, one of each family.

We proceed to determine the directrix curves on S, or what amounts to the same thing, the developables of the directrix congruence of the second kind.

Let u and v increase by δu and δv respectively, where δu and δv are infinitesimals. The point P_y will then move to $P_{y+y_u\delta u+y_v\delta v}$, and the directrix of the second kind of this latter point will join it to the point given by

$$\tau + \tau_u \delta u + \tau_v \delta v,$$

where τ is defined by (129).

We find

$$(130)\qquad \tau_u = Py + Qz + R\rho + S\sigma, \qquad \tau_v = P'y + Q'z + R'\rho + S'\sigma,$$

where we have put

$$(131)\qquad \begin{aligned} &P = a'b_v f + 4a'b^2 g - 2a'bf_v, && Q = 8a'^2b^2 - a'b_{uv} - a'_u b_v,\\ &R = -(2a'bf + 2a'bb_v + ba'_{uu} + b_u a'_u), && S = a'_u b + 2a'b_u, \end{aligned}$$

and

$$(132)\qquad \begin{aligned} &P' = ba'_u g + 4a'^2 bf - 2a'bg_u, && Q' = -(2a'bg + 2a'ba'_u + a'b_{vv} + a'_v b_v),\\ &R' = 8a'^2 b - ba'_{uv} - a'_u b_v, && S' = a'b_v + 2a'_v b. \end{aligned}$$

The directrix of the second kind which belongs to the point $P_{y+y_u\delta u+y_v\delta v}$ joins it to the point $P_{\tau+\tau_u\delta u+\tau_v\delta v}$. An arbitrary point on this directrix will be given by

$$(133)\qquad \lambda(y + y_u\delta u + y_v\delta v) + \mu(\tau + \tau_u\delta u + \tau_v\delta v).$$

The coördinates of such a point will, therefore, be

$$\begin{aligned} &x_1 = \lambda + \mu(P\delta u + P'\delta v), && x_2 = \lambda\delta u + \mu(-a'b_v + Q\delta u + Q'\delta v),\\ &x_3 = \lambda\delta v + \mu(-ba'_u + R\delta u + R'\delta v), && x_4 = \mu(2a'b + S\delta u + S'\delta v). \end{aligned}$$

In order that this point may also be a point of d', its coördinates must satisfy equations (70b), which give the conditions

$$(134)\qquad \begin{aligned} &2b\delta u\lambda + [2b(Q\delta u + Q'\delta v) + b_v(S\delta u + S'\delta v)]\mu = 0,\\ &2a'\delta v\lambda + [2a'(R\delta u + R'\delta v) + a'_u(S\delta u + S'\delta v)]\mu = 0. \end{aligned}$$

whence

$$(135)\qquad \begin{vmatrix} b\delta u, & (2bQ + b_v S)\delta u + (2bQ' + b_v S')\delta v \\ a'\delta v, & (2a'R + a'_u S)\delta u + (2a'R' + a'_u S')\delta v \end{vmatrix} = 0.$$

Let us put

$$(136)\qquad \begin{aligned} L &= 2a'R + a'_u S = -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'^2_u, \\ 2M &= b(2a'R' + a'_u S') - a'(2bQ + b_v S) \\ &= 2a'b(a'b_{uv} - ba'_{uv}) + 2b^2 a'_u a'_v - 2a'^2 b_u b_v, \\ N &= 2bQ' + b_v S' = -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b_v^2. \end{aligned}$$

Then (135) becomes

$$(137)\qquad bL\delta u^2 + 2M\delta u\,\delta v - a'N\delta v^2 = 0.$$

This is the quadratic equation which determines the tangents of the two directrix curves which pass through the point P.

From their presence in equation (137) it is clear that L, M, N must be invariants. In fact, if $M = 0$, the directrix curves will be conjugate curves of the surface; if either L or N vanishes, one of the directrix curves is also an asymptotic line of the surface.

We proceed to express L, M and N in terms of the standard invariants of the first memoir. We showed there that all invariants are functions of

$$A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk,$$

where

$$(138)\qquad \begin{aligned} h &= b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2, \\ k &= a'^2(g + a'_u) - \tfrac{1}{2}a'a'_{vv} + \tfrac{5}{10}a'^2_v, \end{aligned}$$

and of those that could be obtained from these by the operators

$$(139)\qquad U = a'\frac{\partial}{\partial u}, \qquad V = b\frac{\partial}{\partial v},$$

or by the Wronskian process. A rather lengthy calculation gives the following expressions for L, M, and N:

$$(140)\qquad \begin{aligned} L &= \frac{5U(B)^2 - 4a'^2 bU^2(B) - 16a'^6 h}{4a'^4 b}, \\ M &= \frac{a'^3(A,\ V(A)_u) - b^3(B,\ U(B)_v)}{3a'^3 b^3}, \\ N &= \frac{5V(A)^2 - 4a'b^2 V^2(A) - 16b^6 k}{4a'b^4}. \end{aligned}$$

Equation (137) determines the tangents of the directrix curves which pass through P. If the point P moves along either of these curves, the directrix of the second kind d' describes a developable. The cuspidal edges of these two developables intersect d' in two points P_i whose coördinates are given by

$$\lambda_i y + \mu_i \tau \qquad (i=1,2), \tag{141}$$

where, according to (134)

$$\frac{\lambda_i}{\mu_i} = -\frac{2bQ + b_v S + N\dfrac{\delta v_i}{\delta u_i}}{2b} = -\frac{L + (2a'R + a'_u S')\dfrac{\delta v_i}{\delta u_i}}{2a'\dfrac{\delta v_i}{\delta u_i}}, \tag{142}$$

so that λ_1/μ_1 and λ_2/μ_2 are the two roots of the quadratic:

$$\begin{aligned} &-4a'b^2\left(\frac{\lambda}{\mu}\right)^2 - 4b\left[a'(2bQ + b_v S) + M\right]\frac{\lambda}{\mu} \\ &\qquad + bLN - a'(2bQ + b_v S)^2 - 2M(2bQ + b_v S) = 0. \end{aligned} \tag{143}$$

Let us introduce the canonical tetrahedron of reference. The coördinates of the two points P_1 and P_2 are given by (141), so that their coördinates referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ will be

$$x_1 = \lambda_i, \qquad x_2 = -a' b_v \mu_i, \qquad x_3 = -b a'_u \mu_i, \qquad x_4 = 2a' b \mu_i \qquad (i=1,2).$$

According to (97) the non-homogeneous canonical coördinates of these two points will be

$$x = 0, \qquad y = 0, \qquad z = \frac{64a'^2 b^2 \mu_i}{2\lambda_i - a'_u b_v \mu_i} = \frac{1}{K_i} \qquad (i=1,2), \tag{144}$$

where

$$K = \frac{2\lambda - a'_u b_v \mu}{64 a'^2 b^2 \mu} = -\frac{\alpha\beta}{16a'b} + \frac{1}{32a'^2 b^2}\frac{\lambda}{\mu}, \tag{145}$$

whence

$$\frac{\lambda}{\mu} = 2a'b\alpha\beta + 32a'^2 b^2 K, \qquad \alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}. \tag{146}$$

The two values of K are, consequently, the roots of the quadratic equation:

$$\begin{aligned} &2^{10} a'^4 b^4 K^2 + 2^5 a'^2 b^2 \left[\frac{M}{a'b} + 2Q + 2\beta S + 4a'b\alpha\beta\right] K \\ &\qquad + (Q + \beta S + 2a'b\alpha\beta)\left(\frac{M}{a'b} + Q + \beta S + 2a'b\alpha\beta\right) - \frac{LN}{4a'b} = 0. \end{aligned} \tag{147}$$

The coefficients of this equation are invariants. We proceed to express them

in terms of some of those introduced previously. We have (cf. equations (25) and (46)),

$$C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b,$$

$$C' = 8b_{uv} - 8\frac{b_u b_v}{b} - 32a' b^2.$$

If we compute the explicit expression for M, we shall find

$$M = \frac{a'b}{4}(a'C' - bC), \tag{148}$$

and similarly

$$Q + \beta S + 2a' b\alpha\beta = -\tfrac{1}{8}a'C' + 4a'^2 b^2. \tag{149}$$

Consequently the quadratic for K becomes

$$2^{10}a'^4 b^4 K^2 + 2^5 a'^2 b^2 (2a'^2 b^2 - \tfrac{1}{4}bC)K + (4a'^2 b^2 - \tfrac{1}{8}a'C')(4a'^2 b^2 + \tfrac{1}{8}a'C' - \tfrac{1}{4}bC) - \frac{LN}{4a'b} = 0, \tag{150}$$

the invariance of the coefficients being now apparent. Further we find, upon eliminating λ/μ between (142) and (145),

$$K = \frac{C' - 2^5 a' b^2}{2^8 a' b^2} - \frac{N}{2^6 a'^2 b^3}\frac{\delta v}{\delta u}. \tag{151}$$

We may recapitulate as follows. *As the point P of the surface moves along one of the two directrix curves, the directrix of the second kind d' describes one of the two developables of the directrix congruence of which d' is a generator and the ratio $\delta u : \delta v$ will satisfy equation* (137). *The non-homogeneous canonical coördinates of the points P_1 and P_2 of d' in which d' meets the respective edges of regression are*

$$x = y = 0, \qquad z = \frac{1}{K},$$

where K is connected with $\delta v/\delta u$ by (151), *and consequently satisfies the quadratic equation* (150).

Let us consider the congruence made up of the directrices of the first kind d. The line d which belongs to a point P_y of the surface joins the points

$$r = -a'_u y + 2a'z, \qquad s = -b_v y + 2b\rho,$$

and its equations are

$$x_4 = 0, \qquad 2a'bx_1 + ba'_u x_2 + a'b_v x_3 = 0.$$

Let infinitesimal increments δu and δv be given to u and v so that the point P_y moves into $P_{y+y_u\delta u+y_v\delta v}$. The directrix of the first kind which belongs to this new point will be the line joining the point

$$r + r_u\delta u + r_v\delta v = [-a'_u - (a'_{uu} + 2a'f)\delta u - a'_{uv}\delta v]y + (2a' + a'_u\delta u + 2a'_v\delta v)z + (-2a'b\delta u - a'_u\delta v)\rho + 2a'\delta v\cdot\sigma.$$

to the point

$$s + s_u\delta u + s_v\delta v = [-b_v - b_{uv}\delta u - (b_{vv} + 2bg)\delta v]y + (-b_v\delta u - 2a'b\delta v)z + (2b + 2b_u\delta u + b_v\delta v)\rho + 2b\delta u\sigma,$$

The coördinates of any point

$$\lambda'(r + r_u\delta u + r_v\delta v) + \mu'(s + s_u\delta u + s_v\delta v)$$

of this line, will be

$$\begin{aligned} x_1 &= -[a'_u + (a'_{uu} + 2a'f)\delta u + a'_{uv}\delta v]\lambda' - [b_v + b_{uv}\delta u + (b_{vv} + 2bg)\delta v]\mu', \\ x_2 &= (2a' + a'_u\delta u + 2a'_v\delta v)\lambda' - (b_v\delta u + 4a'b\delta v)\mu', \\ x_3 &= -(4a'b\delta u + a'_u\delta v)\lambda' + (2b + 2b_u\delta u + b_v\delta v)\mu', \\ x_4 &= 2a'\delta v\lambda' + 2b\delta u\mu'. \end{aligned} \tag{152}$$

This point will be upon the line d if its coördinates satisfy the equations

$$x_4 = 0, \qquad 2a'bx_1 + ba'_ux_2 + a'b_vx_3 = 0,$$

of d, i. e., if

$$a'\lambda'\delta v + b\mu'\delta u = 0,$$

$$\begin{aligned} &[2a'b^2(2a'f + a'_{uu}) - b^2a'^2_u + 4a'^2b^2b_v]\,\delta u^2 \\ &+ [2a'b(a'_{uv}b - a'b_{uv}) - ba'_u(2a'_vb + b_va') + a'b_v(a'_ub + 2b_ua')]\delta u\delta v \\ &- [2a'^2b(2bg + b_{vv}) + 4a'^2b^2a'_u - a'^2b^2_v]\,\delta v^2 = 0. \end{aligned} \tag{153}$$

It is found, upon computing the coefficients of this quadratic, that it coincides with (137), so that *the curves determined by* (137) *have the same significance for the directrices of the first as for those of the second kind.* In other words, *if a point P moves along a directrix curve of the surface S, both of the directrices of the point P will describe developables at the same time.*

Moreover we have found

$$\frac{\delta u}{\delta v} = -\frac{a'\lambda'}{b\mu'}, \tag{154}$$

so that $\lambda' : \mu'$ will be the two roots of the equation

$$La'\lambda'^2 - 2M\lambda'\mu' - Nb\mu'^2 = 0. \tag{155}$$

Therefore, every directrix of the first kind belongs to two developables of the congruence formed by their totality. The points P'_i where the directrix d, belonging to the point P, intersects the cuspidal edges of the two developables of which it is a generator, are given by

$$\lambda_i' r + \mu_i' s = \lambda_i'(-a_u' y + 2a'z) + \mu_i'(-b_v y + 2b\rho) \qquad (i=1,2),$$

where $\lambda_i' : \mu_i'$ are the two roots of the quadratic (155).

The tangent t_k of one of the directrix curves which passes through P joins this point to the point

$$y + y_u \delta u_k + y_v \delta v_k = y + z\delta u_k + \rho \delta v_k,$$

where $\delta u_k : \delta v_k$ is one of the roots of (137). If l and m are arbitrary quantities, the expression

$$ly + m\delta u_k \cdot z + m\delta v_k \cdot \rho$$

will consequently represent an arbitrary point of this tangent t_k. The point in which t_k intersects d will be obtained if

$$2a'bl + ba_u' m\delta u_k + a'b_v \cdot m\delta v_k = 0,$$

i. e., if

$$\frac{l}{m} = -\frac{ba_u' \delta u_k + a'b_v \delta v_k}{2a'b}.$$

Consequently the point of intersection of t_k and d will be given by

$$-(ba_u' \delta u_k + a'b_v \delta v_k) y + 2a'b(z\delta u_k + \rho \delta v_k) = b\delta u_k \cdot r + a'\delta v_k \cdot s. \tag{156}$$

But the edge of regression of the developable which d describes as P moves along the directrix curve considered, is the locus of the point

$$\lambda_k' r + \mu_k' s,$$

which, according to (154) is the harmonic conjugate of the point (156). We have the following theorem.

As a point P describes a directrix curve of a surface, its directrices of both kinds describe developables. Let the tangent of one of the directrix curves be constructed at the point P, as well as the asymptotic tangents of this point. The harmonic conjugate of the first line with respect to the other two intersects the directrix of the first kind in a point of the cuspidal edge of the developable which it describes.

If the invariants θ and θ' (cf. equations (8) and (14), be expressed in terms of the fundamental invariants, it will be found that

$$\theta = 64L, \qquad \theta' = 64N. \tag{157}$$

We find, therefore, the following result. *If the osculating ruled surfaces of the first kind have flecnode curves with coincident branches, the asymptotic curves of the first kind are also directrix curves.* It is obvious from this remark what will be the state of affairs if θ is zero for particular values of u and v only, instead of vanishing identically.

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